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Some Results on Godel Implication Operator over Bipolar Fuzzy Matrices

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ABSTRACT

At present, bipolar fuzzy sets indeed have robust area of on many fields. Also bipolar fuzzy matrices are admired subject in various wisdoms for example, multi criteria decision making, psychology, qualitative reasoning, etc. Both negative and positive issues to deal with problems live in a society. In this paper, we introduce Godel implication operator on bipolar fuzzy set and also we extend the same to bipolar fuzzy matrices. Some basic algebraic properties like reflexive, transitive, idempotent are obtained and also we give some example. Further, we introduce a new composition using the existing operator min($\wedge$) along with Go ̈del implication on bipolar fuzzy matrices. Various results of this new composition are discussed.

Keywords*:*Bipolar fuzzy sets; Bipolar fuzzy matrices; Reflexive; Transitive; Idempotent

1. Introduction

Zadeh [1] has introduced the idea of fuzzy sets in 1965, fuzzy concepts developed in about all fields. Thomason [2] has first, introduced the concept of fuzzy matrices in 1977. The elements of fuzzy matrices getting values in the closed interval [0,1]. Zhang [3] first, introduced by concept of bipolar fuzzy set. After, has bipolar fuzzy logic interval based is defined and presented in fuzzy number based on bipolar fuzzy logic. Madhumangal Pal and Sanjib Mondal [4] has introduced bipolar fuzzy matrices in 2019. An order ($\leq$) is defined on bipolar fuzzy algebra and bipolar fuzzy relations. Some results on bipolar fuzzy matrices are discussed. Gupta and Kiszka [5] are discussed Gödel implication operators for fuzzy logic control in 1989. The upper bounds for a multivariable fuzzy logic including this operator are examined. Hashimoto [6] has defined reflexive and transitive on square matrix over decomposition of fuzzy matrices. Murugadas and Lalitha [7] has discussed bi implication operator ($\leftrightarrow$) into intuitionistic fuzzy set and some properties are verified. Emam and Fndh [8] has defined the bifuzzy matrices of min-max and max-min compositions. Sanjib Mondal and M.Pal [9] has defined an idempotent, diagonally dominant of bipolar fuzzy matrices. Moreover, eigen vector and eigen values for bipolar fuzzy matrices are investigated. Li, pinke and Jin Qingwel [10] has investigated bipolar max-min composition over fuzzy relational equations. Xiao-peng yang [11] has introduced max-lukasiewicz composition on bipolar fuzzy relation equations system. In [12], Ghous Ali et al. studied bipolar fuzzy relation decision systems and bipolar fuzzy relation systems. In [13], Muthuraji et al. studied the bounded sum and bounded product on bipolar fuzzy matrices by deriving algebraic results. Finally, they constructed a commutative monoid on bounded sum and bounded product over bipolar fuzzy matrices. In [14], Muthuraji and Punitha discussed the algebraic sum and algebraic product over bipolar fuzzy sets and bipolar fuzzy matrices. Naime Demirtas and Orhan Dalkilic [15] presented the application of this algorithm in the medical field was illustrated through practical examples, demonstrating its potential to improve diagnostic processes and decision-making in medical practice.

Hashimoto [16] has investigated definition of Gödel implication operator in the ordinary fuzzy matrices in 1984. Basic properties are obtained by this operator. Emam [17] has defined Gödel implication operator on intuitionistic fuzzy matrices in 2020. Using this operator, some results on intuitionistic fuzzy matrices are obtained. In this paper, we introduce Gödel implication operator on bipolar fuzzy set and also we extend the same to bipolar fuzzy matrices. Some basic algebraic properties are discussed in reflexive, transitive etc. Further, we introduce a new composition operator using the existing operator min($\wedge$) along with Gödel implication operator. Several results of this new composition are discussed.

2. Preliminaries

We remember some relevant basic definitions and results will be used later.

Definition 2.1 [16] The definition of fuzzy set is given by the membership function $\mu_F: U \to [0,1]$ elements of the universe of discourse U can belong to the fuzzy set with any value between 0 and 1. The degree of membership of an element u, $0 \le \mu_F(u) \le 1$.

Definition 2.2 [14] let X be a matrix, $X = \left[(x_{ij})\right]_{p\times q}$, where $x_{ij} \in [0,1], 1 \le i \le p$ and $1 \le j \le q$ then X is called fuzzy matrix.

Definition 2.3 [8,17] A bipolar fuzzy set is a pair of $(-x_n, x_p)$ where $-x_n: x \to [-1,0]$ and $x_p: x \to [0,1]$ are the respectively negative and positive membership degree of $x \in X$. The set of all bipolar fuzzy set on X is denoted by $B_F(X)$. Bipolar fuzzy set is an extension of fuzzy set.

Definition 2.4 [8,10] A bipolar fuzzy matrix $X = \left[(x_{ij})\right]$ where (x_{ij}) is defined as $\langle -x_{ijn}, x_{ijp}\rangle$ whose x_{ijn}, $x_{ijp} \in [0,1]$ $\forall\, i, j$.

Definition 2.5 (Some special bipolar fuzzy matrices) [10]

(i) A zero bipolar fuzzy matrix is the arrangement of zero elements into rows and columns. It is a matrix with all its entries equal to zero. A matrix whose entries are all zero O_n of order $n \times n$ is the matrix of elements $O_b = (0,0)$.

(ii) A Identity bipolar fuzzy matrix I_n is an $n \times n$ matrix with the main diagonal elements are $i_b = \langle -1, 1\rangle$ and all order elements are $O_b = (0,0)$.

(iii) Let J_n be a unit bipolar fuzzy matrix whose all the entries of J_n of order $n \times n$ is the matrix of elements $\langle -1, 1\rangle$.

Definition 2.6 (Operations on bipolar fuzzy matrix) [1,10] Let $X = [x_{ij} = \langle -x_{ijn}, x_{ijp}\rangle]_{p\times q}$, $Y = [y_{ij} = \langle -y_{ijn}, y_{ijp}\rangle]_{p\times q}$ and $Z = [z_{ij} = \langle -z_{ijn}, z_{ijp}\rangle]_{q\times r}$ be three bipolar fuzzy matrices.

(i) $X \vee Y = X + Y = (x_{ij} + y_{ij})_{p\times q} = (-\max\{x_{ijn,}y_{ijn}\}, max\{x_{ijp,}y_{ijp}\})_{p\times q}\ \forall\, i, j$

(ii) $X \wedge Y = X \cdot Y = (x_{ij} \cdot y_{ij})_{p\times q} = (-\min\{x_{ijn,}y_{ijn}\}, min\{x_{ijp,}y_{ijp}\})_{p\times q}\ \forall\, i, j$

(iii) $X^t = [(x_{ji})]$ (transpose of X),

(iv) $R = XZ = [r_{ij} = \left\langle -r_{ijn}, r_{ijp}\right\rangle = \left\langle -\overset{n}{\underset{t=1}{\vee}}(x_{itn} \wedge y_{tjn}), \overset{n}{\underset{t=1}{\vee}}(x_{itp} \wedge y_{tjp})\right\rangle]_{p\times r,}$

(v) $X \le Y$ iff $x_{ijn} \le y_{ijn}$ and $x_{ijp} \le y_{ijp}$ to each $i = 1,2, \dots, p, j = 1,2, \dots, q$.

Definition 2.7 [2,6] An $q \times q$ bipolar fuzzy matrix X is called reflexive iff $x_{ii} = (-1,1)$ to each $i \le q$.

Definition 2.8 [2,6] An $q \times q$ bipolar fuzzy matrix X is called transitive iff $X^2 \le X$ and said to be impotent iff $X^2 = X$.

Definition 2.9 [5] The definition of the Gödel implication operator in the ordinary fuzzy case defined as follows:

$$x \to y = \begin{cases} 1 & if \quad x \le y, \\ y & if \quad x > y. \end{cases} \qquad \text{to each } x, y \in [0,1].$$

The operator $\to$ is the Gödel implication operator which is well known in many chapters of fuzzy mathematics.

3. Methodology

Reflexive and transitive on bipolar fuzzy matrices

In this part, we introduce the Gödel implication operator on bipolar fuzzy set and bipolar fuzzy matrices. Further, some results are discussed on bipolar fuzzy set and bipolar fuzzy matrices. We are investigated reflexive and transitive bipolar fuzzy matrices on Gödel implication operator.

Definition 3.1 For $x = \langle -x_n, x_p\rangle$, $y = \langle -y_n, y_p\rangle \in B_F(S)$, we define $x \overset{B_F}{\to} y$ as,

$$x \overset{B_F}{\to} y = \begin{cases} \langle -1,1\rangle & when\ x_n \le y_n, \\ \langle -y_n, y_p\rangle & when\ x_n > y_n, x_p > y_p, \\ \langle -y_n, 0\rangle & when\ x_n > y_n, x_p \le y_p. \end{cases}$$

This operator $\overset{B_F}{\to}$ is called the Gödel implication operator on the bipolar fuzzy set.

Moreover, $x \overset{B_F}{\to} y = y \overset{B_F}{\leftarrow} x$.

Definition 3.2 For $X = \langle -x_{ijn}, x_{ijp}\rangle_{p\times q} Y = \langle -y_{ijn}, y_{ijp}\rangle_{p\times q} \in B_F(M)$, we define $X \overset{B_F}{\to} Y$ as,

$$X \overset{B_F}{\rightarrow} Y = \begin{cases} \langle -1,1 \rangle & when\ x_{ijn} \le y_{ijn}, \\ \langle -y_{ijn}, y_{ijp} \rangle & when\ x_{ijn} > y_{ijn},\ x_{ijp} > y_{ijp}, \\ \langle -y_{ijn}, 0 \rangle & when\ x_{ijn} > y_{ijn}, x_{ijp} \le y_{ijp}. \end{cases} \quad \text{When } 1 \le i \le p, 1 \le j \le q.$$

This operator $\overset{B_F}{\rightarrow}$ is called the Gödel implication operator on the bipolar fuzzy matrices.

Lemma 3.1 For $x = \langle -x_n, x_p \rangle$, $y = \langle -y_n, y_p \rangle$, $z = \langle -z_n, z_p \rangle \in B_F(S)$, we have $(x \overset{B_F}{\rightarrow} y) \overset{B_F}{\leftarrow} z = x \overset{B_F}{\rightarrow} (y \overset{B_F}{\leftarrow} z) = y \overset{B_F}{\leftarrow} (x \wedge z)$.

Proof: Given the definition of the operator $\overset{B_F}{\rightarrow}$ on the set B_F, we have following three cases everyone has also three subcases. The cases are given by:

Case (a)$x_n \le y_n$,

Case (b)$x_n > y_n, x_p > y_p$,

Case (c)$x_n > y_n, x_p \le y_p$.

Now let us prove that lemma for **Case (c)** andother two cases are similar to **Case (c)**$x_n > y_n, x_p \le y_p$.

Subcase (i): If $z_n \le y_n$, then

$$(\langle -x_n, x_p \rangle \overset{B_F}{\rightarrow} \langle -y_n, y_p \rangle) \overset{B_F}{\leftarrow} \langle -z_n, z_p \rangle = \langle -y_n, 0 \rangle \overset{B_F}{\leftarrow} \langle -z_n, z_p \rangle = \langle -1, 1 \rangle$$
$$\langle -x_n, x_p \rangle \overset{B_F}{\rightarrow} (\langle -y_n, y_p \rangle) \overset{B_F}{\leftarrow} \langle -z_n, z_p \rangle) = \langle -x_n, x_p \rangle \overset{B_F}{\rightarrow} \langle -1, 1 \rangle = \langle -1, 1 \rangle$$

Since, $x_p \wedge z_p \le y_p$, we obtain

$$\langle -y_n, y_p \rangle \overset{B_F}{\leftarrow} (\langle -x_n, x_p \rangle \ \wedge \langle -z_n, z_p \rangle) = \langle -1, 1 \rangle.$$

Subcase (ii): If $z_n > y_n, z_p > y_p$, then

$$(\langle -x_n, x_p \rangle \overset{B_F}{\rightarrow} \langle -y_n, y_p \rangle) \overset{B_F}{\leftarrow} \langle -z_n, z_p \rangle = \langle -y_n, 0 \rangle \overset{B_F}{\leftarrow} \langle -z_n, z_p \rangle = \langle -y_n, 0 \rangle$$
$$\langle -x_n, x_p \rangle \overset{B_F}{\rightarrow} (\langle -y_n, y_p \rangle) \overset{B_F}{\leftarrow} \langle -z_n, z_p \rangle) = \langle -x_n, x_p \rangle \overset{B_F}{\rightarrow} \langle -y_n, y_p \rangle = \langle -y_n, 0 \rangle$$

Since,$x_n \wedge z_n > y_n$ and $x_p \vee z_p \le y_p$ we obtain

$$\langle -y_n, y_p \rangle \overset{B_F}{\leftarrow} \left(\langle -x_n, x_p \rangle \ \wedge \langle -z_n, z_p \rangle\right) = \langle -y_n, 0 \rangle.$$

Subcase (iii): If, $z_n > y_n, z_p \le y_p$, then

$$(\langle -x_n, x_p \rangle \overset{B_F}{\rightarrow} \langle -y_n, y_p \rangle) \overset{B_F}{\leftarrow} \langle -z_n, z_p \rangle = \langle -y_n, 0 \rangle \overset{B_F}{\leftarrow} \langle -z_n, z_p \rangle = \langle -y_n, 0 \rangle$$
$$\langle -x_n, x_p \rangle \overset{B_F}{\rightarrow} (\langle -y_n, y_p \rangle) \overset{B_F}{\leftarrow} \langle -z_n, z_p \rangle) = \langle -x_n, x_p \rangle \overset{B_F}{\rightarrow} \langle -y_n, 0 \rangle = \langle -y_n, 0 \rangle$$

Since, $x_n \wedge z_n > y_n$ and $x_p \vee z_p \le y_p$ we obtain

$$\langle -y_n, y_p \rangle \overset{B_F}{\leftarrow} \left(\langle -x_n, x_p \rangle \ \wedge \langle -z_n, z_p \rangle\right) = \langle -y_n, 0 \rangle.$$

Hence, through all cases we conclude

$$(x \overset{B_F}{\rightarrow} y) \overset{B_F}{\leftarrow} z = x \overset{B_F}{\rightarrow} (y \overset{B_F}{\leftarrow} z) = y \overset{B_F}{\leftarrow} (x \wedge z) \text{ for each } x, y, z \in B_F(S).$$

Through this lemma, we can correspond $x \overset{B_F}{\rightarrow} y \overset{B_F}{\leftarrow} z$ in place of $(x \overset{B_F}{\rightarrow} y) \overset{B_F}{\leftarrow} z$ or $x \overset{B_F}{\rightarrow} (y \overset{B_F}{\leftarrow} z)$. we may takeout enclosure. Also, we consider that this lemma is similar to the connection

$$(x \wedge z) \overset{B_F}{\rightarrow} y = x \overset{B_F}{\rightarrow} (\mathrm{z} \overset{B_F}{\rightarrow} \mathrm{y}) = z \overset{B_F}{\rightarrow} (x \overset{B_F}{\rightarrow} \mathrm{y}).$$

Lemma 3.2 For $x, y, z, s \in B_F(S)$, we get $y \overset{B_F}{\rightarrow} x \overset{B_F}{\leftarrow} z < s$ implies $x < s, x < y$ and $x < z$.

Proof. Let $y \overset{B_F}{\rightarrow} x \overset{B_F}{\leftarrow} z < s$. Then we get $x \le y \overset{B_F}{\rightarrow} x \le (y \overset{B_F}{\rightarrow} x) \overset{B_F}{\leftarrow} z = y \overset{B_F}{\rightarrow} x \overset{B_F}{\leftarrow} z < s$. Thus, $x < s$ and $y \overset{B_F}{\rightarrow} x < s$ which means $x < y$ and so

$$(y \overset{B_F}{\rightarrow} x) \overset{B_F}{\leftarrow} z = x \overset{B_F}{\leftarrow} z < s \text{ which means } x < z.$$

Lemma 3.3 For $x, y, z \in B_F(S)$, if we have $x_p \wedge z_p \le y_p$, then $x \overset{B_F}{\rightarrow} y \overset{B_F}{\leftarrow} z = 1$.

Proof. It is clear from 3.1.

From lemma 3.1, 3.2, 3.3 we have the following theorems.

Theorem 3.1 For $X = \langle -x_{ijn}, x_{ijp} \rangle$, $Y = \langle -y_{ijn}, y_{ijp} \rangle$,$Z = \langle -z_{ijn}, z_{ijp} \rangle \in B_F(M)$, we have $(X \overset{B_F}{\rightarrow} Y) \overset{B_F}{\leftarrow} Z = X \overset{B_F}{\rightarrow} \left(Y \overset{B_F}{\leftarrow} Z\right) = Y \overset{B_F}{\leftarrow} (X \wedge Z)$.

Theorem 3.2 For$X, Y, Z, S \in B_F(M)$, we get $Y \overset{B_F}{\rightarrow} X \overset{B_F}{\leftarrow} Z < S$ implies $X < S, X < Y$ and $X < Z$.

Theorem 3.3 For $X, Y, Z \in B_F(M)$, if we have $x_{ijp} \wedge z_{ijp} \le y_{ijp}$, then $X \overset{B_F}{\to} Y \overset{B_F}{\leftarrow} Z = 1$.
Proposition 3.1 If $X \in BFM$ is reflexive and transitive, then X is idempotent.
Proof: Given X is transitive, it is sufficient to show that, $X^2 \ge X$. Let X be kind of $q \times q$ and let $R = X^2$. Then

$$\begin{aligned} r_{ij} = \langle -r_{ijn}, r_{ijp} \rangle &= \left\langle -\bigvee_{k=1}^{q}(x_{ikn} \wedge x_{kjn}), \bigvee_{k=1}^{q}(x_{ikp} \wedge x_{kjp}) \right\rangle \\ &\ge \langle x_{ijn} \wedge x_{jjn}, x_{ijp} \wedge x_{jjp} \rangle \\ &= \langle x_{ijn} \wedge 1, x_{ijp} \wedge 1 \rangle \\ &= \langle -x_{ijn}, x_{ijp} \rangle \\ &= (x_{ij}).\ \text{(since } X \text{ is reflexive)} \end{aligned}$$

Therefore, $X^2 \ge X$ and X is an idempotent.
Theorem 3.4 Let X and Y be two $p \times q$ bipolar fuzzy matrices thus $X \le Y$. Then $X \overset{B_F}{\leftarrow} Y^t$ and $Y^t \overset{B_F}{\to} X$ are reflexive and transitive.
Proof: Let $S = X \overset{B_F}{\leftarrow} Y^t$. We prove S is transitive, we will must show that $s_{ij} \ge s_{il} \wedge s_{lj}$ for each $l \le q$. Suppose that $s_{il} \wedge s_{lj} = z > 0$. Then

$$s_{il} = \bigwedge_{t=1}^{q}(x_{it} \xleftarrow{B_F} y_{lt}) \ge z, s_{lj} = \bigwedge_{t=1}^{q}(x_{lt} \xleftarrow{B_F} y_{jt}) \ge z$$

If, $s_{ij} = \langle -s_{ijn}, s_{ijp} \rangle$ such that $s_{ijn} < z_n$ and $s_{ijp} > z_p$ where $z = \langle -z_n, z_p \rangle$, that is, if $\langle -x_{ihn}, x_{ihp} \rangle \overset{B_F}{\leftarrow} \langle -y_{jhn}, y_{jhp} \rangle < \langle -z_n, z_p \rangle$. Then $x_{ihn} < z_n$, $x_{ihp} > z_p$ and also $x_{ihn} < y_{jhn}$ and $x_{ihp} > y_{jhp}$
Since $s_{il} \ge z$, we have $\langle -x_{ihn}, x_{ihp} \rangle \overset{B_F}{\leftarrow} \langle -y_{lhn}, y_{lhp} \rangle \ge \langle -z_n, z_p \rangle$. But $x_{ihn} < z_n$, $x_{ihp} > z_p$. Then $x_{ihn} \ge y_{lhn}$ and $x_{ihp} \le y_{lhp}$.
Moreover, since we have $s_{lj} \ge z$, $\langle -x_{lhn}, x_{lhp} \rangle \overset{B_F}{\leftarrow} \langle -y_{jhn}, y_{jhp} \rangle \ge \langle -z_n, z_p \rangle$ therefore that $x_{lhn} \ge y_{jhn}$ and $x_{lhp} \le y_{jhp}$. Since, $X \le Y$, we have $z_n > x_{ihn} \ge y_{lhn} \ge x_{lhn} \ge y_{jhn}$. However, it is a contradiction. So that $s_{ijn} \ge z_n$. Also, $z_p < x_{ihp} \le y_{lhp} \le x_{lhp} \le y_{jhp}$ it is a contradiction. So, $s_{ijp} \le z_p$.
Hence, $s_{ij} = \langle -s_{ijn}, s_{ijp} \rangle \ge \langle -z_n, z_p \rangle = z$ and S is transitive. The reflexive of S is obviously by the definition of the operator $\overset{B_F}{\leftarrow}$. By a similarly we can prove that $Y^t \overset{B_F}{\to} X$ is reflexive and transitive.
Corollary 3.1 For any $p \times q$ bipolar fuzzy matrix $X, X \overset{B_F}{\leftarrow} X^t$ and $X^t \overset{B_F}{\to} X$ are idempotent.
Proof: By theorem 3.4 and proposition 3.1.
Theorem 3.5 Let X be any $q \times q$ bipolar fuzzy matrix. Then X is reflexive and transitive iff $X \overset{B_F}{\leftarrow} X^t = X$ (moreover iff $X^t \overset{B_F}{\to} X = X$).
Proof: Assume that X is reflexive and transitive and take in $S = X \overset{B_F}{\leftarrow} X^t$. Then,

$$s_{ij} = \bigwedge_{l=1}^{q}(\langle -x_{iln}, x_{ilp} \rangle \xleftarrow{B_F} \langle -x_{jln}, x_{jlp} \rangle).$$

Part (a) Assume $s_{ij} = \langle -s_{ijn}, s_{ijp} \rangle = \langle -z_n, z_p \rangle > \langle 0, 0 \rangle$ .Then $\langle -x_{ijn}, x_{ijp} \rangle \overset{B_F}{\leftarrow} \langle -x_{jjn}, x_{jjp} \rangle \ge \langle -z_n, z_p \rangle$. ie) $\langle -x_{ijn}, x_{ijp} \rangle \overset{B_F}{\leftarrow} \langle -1,1 \rangle \ge \langle -z_n, z_p \rangle$ which means $x_{ij} = \langle -x_{ijn}, x_{ijp} \rangle \ge \langle -z_n, z_p \rangle$

$= s_{ij}$ and so $X \ge X \overset{B_F}{\leftarrow} X^t$.

Part (b) Assume that $x_{ij} = \langle -x_{ijn}, x_{ijp} \rangle = \langle -z_n, z_p \rangle > \langle 0, 0 \rangle$ and $s_{ij} = \langle -s_{ijn}, s_{ijp} \rangle = \langle -x_{ikn}, x_{ikp} \rangle \overset{B_F}{\leftarrow} \langle -x_{jkn}, x_{jkp} \rangle$ for some $k \le q$.
Definition on Gödel implication operator ($\overset{B_F}{\leftarrow}$), accept the following cases:
Case (1) If, $\langle -x_{ikn}, x_{ikp} \rangle \overset{B_F}{\leftarrow} \langle -x_{jkn}, x_{jkp} \rangle = \langle -1,1 \rangle$, then $s_{ij} = \langle -s_{ijn}, s_{ijp} \rangle \ge \langle -z_n, z_p \rangle = z$
Case (2) If, $\langle -x_{ikn}, x_{ikp} \rangle \overset{B_F}{\leftarrow} \langle -x_{jkn}, x_{jkp} \rangle = \langle -x_{ikn}, x_{ikp} \rangle < \langle -z_n, z_p \rangle$ then $x_{ikn} < x_{jkn}$, $x_{ikp} > x_{jkp}$, $x_{ikn} < z_n$ and $x_{ikp} > z_p$ .So that, $\langle -x_{ijn}, x_{ijp} \rangle \wedge \langle -x_{jkn}, x_{jkp} \rangle \le \langle -x_{ikn}, x_{ikp} \rangle$ (using transitive of X) thus, $\langle -z_n, z_p \rangle \wedge \langle -x_{jkn}, x_{jkp} \rangle \le \langle -x_{ikn}, x_{ikp} \rangle$. Therefore, $x_{ikn} < z_n \wedge x_{jkn} \le x_{ikn}$ and $x_{ikp} > z_p \vee x_{jkp} \ge x_{ikp}$. However, there are contradictions and $s_{ijn} \ge z_n$ and $s_{ijp} \le z_p$. Then $s_{ijn} \ge z$.

Case (3) If $s_{ij} = \langle -x_{ikn}, 0\rangle$then $x_{ikn} < x_{jkn}$ as we have $0 < z_p$, it is sufficient to show that $x_{ikn} \geq z_n$ Suppose that $x_{ikn} < z_n$. We have case (2) say as contradiction and $x_{ikn} \geq z_n$ and $s_{ij} \geq z$. From all cases we obtain $X \overset{B_F}{\leftarrow} X^t \geq X$, from (a) and (b) we have $X \overset{B_F}{\leftarrow} X^t = X$.

Converse, If $X \overset{B_F}{\leftarrow} X^t = X$,then by corollary 3.1 X is reflexive and transitive and so idempotent. Similar, we can prove that $X^t \overset{B_F}{\rightarrow} X = X$. However , there are contradictions and $s_{ijn} \geq z_n$ and $s_{ijp} \leq z_p$. thus $s_{ij} \geq z$.

Corollary 3.2 If X is reflexive and transitive on bipolar fuzzy matrix, then $(X \overset{B_F}{\leftarrow} X^t)X = (X^t \overset{B_F}{\rightarrow} X)X = X$.

Proof: At theorem 3.5 and corollary 3.1 .

Corollary 3.3 If X is a reflexive and transitive bipolar fuzzy matrix, then $(X^t \overset{B_F}{\rightarrow} X) \overset{B_F}{\leftarrow} X^t = X$.

Theorem 3.5 shows absorbing properties of pre-orders. Thus X is a matrix constitute a pre-orders iff $X \overset{B_F}{\leftarrow} X^t = X$ (or) $X^t \overset{B_F}{\rightarrow} X = X$. However, since $X \overset{B_F}{\leftarrow} X^t$ is get in by using X if we multiple $X \overset{B_F}{\leftarrow} X^t$ by X. That is product $(X \overset{B_F}{\leftarrow} X^t)X$ is equal to X.

Example 3.1 Let

$$X = \begin{bmatrix} \langle -0.7, 0.3\rangle & \langle -0.8, 0.2\rangle \\ \langle -0.6, 0.4\rangle & \langle -0.9, 0\rangle \\ \langle -0.9, 0.1\rangle & \langle -0.4, 0.6\rangle \end{bmatrix}$$

Then, $$X \overset{B_F}{\leftarrow} X^t = \begin{bmatrix} \langle -0.7, 0.3\rangle & \langle -0.8, 0.2\rangle \\ \langle -0.6, 0.4\rangle & \langle -0.9, 0\rangle \\ \langle -0.9, 0.1\rangle & \langle -0.4, 0.6\rangle \end{bmatrix} \overset{B_F}{\leftarrow} \begin{bmatrix} \langle -0.7, 0.3\rangle & \langle -0.6, 0.4\rangle & \langle -0.9, 0.1\rangle \\ \langle -0.8, 0.2\rangle & \langle -0.9, 0\rangle & \langle -0.4, 0.6\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle -1, 1\rangle & \langle -0.8, 0\rangle & \langle -0.7, 0\rangle \\ \langle -0.6, 0\rangle & \langle -1, 1\rangle & \langle -0.6, 0\rangle \\ \langle -0.4, 0\rangle & \langle -0.4, 0\rangle & \langle -1, 1\rangle \end{bmatrix}$$

It is clear that $X \overset{B_F}{\leftarrow} X^t$ is reflexive and

$$\left(X \overset{B_F}{\leftarrow} X^t\right)^2 = \begin{bmatrix} \langle -1, 1\rangle & \langle -0.8, 0\rangle & \langle -0.7, 0\rangle \\ \langle -0.6, 0\rangle & \langle -1, 1\rangle & \langle -0.6, 0\rangle \\ \langle -0.4, 0\rangle & \langle -0.4, 0\rangle & \langle -1, 1\rangle \end{bmatrix} \begin{bmatrix} \langle -1, 1\rangle & \langle -0.8, 0\rangle & \langle -0.7, 0\rangle \\ \langle -0.6, 0\rangle & \langle -1, 1\rangle & \langle -0.6, 0\rangle \\ \langle -0.4, 0\rangle & \langle -0.4, 0\rangle & \langle -1, 1\rangle \end{bmatrix}$$

$$= X \overset{B_F}{\leftarrow} X^t.$$

That is $X \overset{B_F}{\leftarrow} X^t$ is reflexive and transitive and so idempotent. Further, we let $Y = X \overset{B_F}{\leftarrow} X^t$

$$Y \overset{B_F}{\leftarrow} Y^t = \begin{bmatrix} \langle -1, 1\rangle & \langle -0.8, 0\rangle & \langle -0.7, 0\rangle \\ \langle -0.6, 0\rangle & \langle -1, 1\rangle & \langle -0.6, 0\rangle \\ \langle -0.4, 0\rangle & \langle -0.4, 0\rangle & \langle -1, 1\rangle \end{bmatrix} \overset{B_F}{\leftarrow} \begin{bmatrix} \langle -1, 1\rangle & \langle -0.6, 0\rangle & \langle -0.4, 0\rangle \\ \langle -0.8, 0\rangle & \langle -1, 1\rangle & \langle -0.4, 0\rangle \\ \langle -0.7, 0\rangle & \langle -0.6, 0\rangle & \langle -1, 1\rangle \end{bmatrix} = Y$$

$$\left(Y \overset{B_F}{\leftarrow} Y^t\right) Y = \begin{bmatrix} \langle -1, 1\rangle & \langle -0.8, 0\rangle & \langle -0.7, 0\rangle \\ \langle -0.6, 0\rangle & \langle -1, 1\rangle & \langle -0.6, 0\rangle \\ \langle -0.4, 0\rangle & \langle -0.4, 0\rangle & \langle -1, 1\rangle \end{bmatrix} \begin{bmatrix} \langle -1, 1\rangle & \langle -0.8, 0\rangle & \langle -0.7, 0\rangle \\ \langle -0.6, 0\rangle & \langle -1, 1\rangle & \langle -0.6, 0\rangle \\ \langle -0.4, 0\rangle & \langle -0.4, 0\rangle & \langle -1, 1\rangle \end{bmatrix} = Y$$

4. Results on a new composition operator

In this section, we define a new composition using the existing operator min($\wedge$) along with Gödel implication operator. Various results of this composition are discussed.

Definition 4.1 The new composition operator of two bipolar fuzzy matrices $X = [x_{ij} = \langle -x_{ijn}, x_{ijp}\rangle]_{p\times q}$ and $Y = [y_{ij} = \langle -y_{ijn}, y_{ijp}\rangle]_{q\times r}$ defined as $Z = X \overset{B_F}{\rightarrow} Y$ such that $z_{ij} = \overset{q}{\underset{k=1}{\wedge}} (x_{ik} \xrightarrow{B_F} y_{kj})$.

Proposition 4.1 Let $X = [x_{ij}]_{m\times t}$, $Y = [y_{ij}]_{t\times s}$ and $Z = [z_{ij}]_{s\times n}$ are three bipolar fuzzy matrices. Then $(X \overset{B_F}{\rightarrow} Y) \overset{B_F}{\leftarrow} Z = X \overset{B_F}{\rightarrow} (Y \overset{B_F}{\leftarrow} Z)$.

Proof: Let $A = (X \overset{B_F}{\rightarrow} Y) \overset{B_F}{\leftarrow} Z$ and $B = X \overset{B_F}{\rightarrow} (Y \overset{B_F}{\leftarrow} Z)$ then using by lemma 3.1

$$a_{ij} = \bigwedge_{p=1}^{s} \left[\bigwedge_{q=1}^{t} (x_{iq} \overset{B_F}{\rightarrow} y_{qp}) \overset{B_F}{\leftarrow} z_{pj} \right]$$

$$= \bigwedge_{p=1}^{s} \bigwedge_{q=1}^{t} \left(x_{iq} \overset{B_F}{\rightarrow} y_{qp} \overset{B_F}{\leftarrow} z_{pj} \right) \qquad and$$

$$b_{ij} = \bigwedge_{q=1}^{t} \left[x_{iq} \overset{B_F}{\rightarrow} \bigwedge_{p=1}^{s} (y_{qp} \overset{B_F}{\leftarrow} z_{pj}) \right]$$

$$= \bigwedge_{p=1}^{s} \bigwedge_{q=1}^{t} \left(x_{iq} \overset{B_F}{\rightarrow} y_{qp} \overset{B_F}{\leftarrow} z_{pj} \right).$$

Hence, $a_{ij} = b_{ij}$

For, this is proposition we represent $(X \overset{B_F}{\rightarrow} Y) \overset{B_F}{\leftarrow} Z$ (or) $X \overset{B_F}{\rightarrow} (Y \overset{B_F}{\leftarrow} Z)$ by $X \overset{B_F}{\rightarrow} Y \overset{B_F}{\leftarrow} Z$.

Proposition 4.2 For any $p \times q$ bipolar fuzzy matrix X,

(i) $X \overset{B_F}{\leftarrow} J_q = X$,

(ii) $J_p \overset{B_F}{\rightarrow} X = X$.

Proof. (i) Let $Y = X \overset{B_F}{\leftarrow} J_q$. Then $y_{ij} = \overset{q}{\underset{l=1}{\wedge}} (x_{il} \xleftarrow{B_F} \delta_{lj}) = x_{ij} \overset{B_F}{\leftarrow} \delta_{jj} = x_{ij} \overset{B_F}{\leftarrow} 1 = x_{ij}$. Therefore $Y = X$.

(ii) Similarl to (i).

5. Conclusion

Bipolar is an interesting topic in some domains. In this topic, we defined Gödel implication operator on bipolar fuzzy set and bipolar fuzzy matrices. Some results are proved for reflexive, transitive and idempotent on BFMs. Also, defined a new composition operator on BFMs. Few properties like unity and distributive are illustrated on BFMs. Thus, Gödel implication operator on bipolar fuzzy matrices are very helpful for moreover studies.

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Declarations

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