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# The Regulatory Principles Involved in the New Neutrosophic Integrals and Their Operational Outcomes

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ABSTRACT

Classical mathematical analysis is built on precise and deterministic concepts, where quantities, limits, and operations are assumed to be exactly known. However, many real-life phenomena involve uncertainty, vagueness, indeterminacy, and incomplete information, which cannot be adequately handled using classical or even fuzzy mathematical frameworks alone. The main aim of this study is to examine the concept of improper neutrosophic integrals, with emphasis on their definitions, types, and convergence properties. Specifically, this study is to present the foundational concepts of neutrosophic logic and neutrosophic calculus, investigate conditions for convergence and divergence of improper neutrosophic integrals, solve illustrative examples involving different classes of neutrosophic functions, and finally, to compare improper neutrosophic integrals with their classical counterparts.

**Keywords***:*Neutrosophic integrals; Neutrosophic functions; Improper neutrosophic integrals; Classical counterparts; Uncertainty

## 1. Introduction

Many real-life phenomena involve uncertainty, vagueness, indeterminacy, and incomplete information, which cannot be adequately handled using classical or even fuzzy mathematical frameworks alone.(Please, see [ 1 - 7] ) In response to these limitations**,** neutrosophic logic, introduced by Florentin Smarandache, provides a more general and flexible mathematical structure for modeling indeterminate information. Neutrosophic calculus extends classical calculus by incorporating indeterminacy into fundamental concepts such as limits, continuity, differentiation, and integration. One important aspect of calculus is the study of improper integrals, which arises when the interval of integration is infinite or when the integrand becomes unbounded. Improper integrals play a crucial role in mathematical analysis, physics, probability theory, and engineering. This particular work focuses on the extension of improper integrals to the neutrosophic setting, leading to the concept of improper neutrosophic integrals. The study examines their definitions, classifications, convergence and divergence properties, as well as their illustrative examples, thereby contributing to the growth as well as substantial developments in the research activities in the entire concepts of the neutrosophic calculus.(Please, see [ 8 - 12] )

### *1.1 Background of Neutrosophic Logic*

Neutrosophic logic was proposed as a generalization of classical logic, fuzzy logic, and intuitionistic fuzzy logic. Unlike classical logic, which is based on binary truth values (true or false), neutrosophic logic characterizes information using three independent components: truth (T), indeterminacy (I), and falsity (F). These components are not necessarily dependent on each other and can take values beyond the classical unit interval. Building on neutrosophic logic, the concept of neutrosophic setswas introduced to handle uncertainty in mathematical structures. This development later led to neutrosophic numbers and neutrosophic functions, which serve as the foundation of neutrosophic calculus. In particular, neutrosophic real numbers allow the representation of quantities that contain an indeterminate part, making them suitable for modeling real-world problems involving incomplete or ambiguous data. Neutrosophic calculus extends classical calculus by redefining limits, continuity, derivatives, and integrals in the presence of indeterminacy. Recent studies have explored definite and indefinite neutrosophic integrals, integration techniques, and applications. The study of improper neutrosophic integrals is a natural continuation of this development, addressing cases where the interval of integration is infinite or the integrand is undefined at certain points. We implore our esteemed readers to also consult [ 10 to 18] of our references as well.

***1.2 Definition of Basic Terms***

For clarity and consistency some basic terms used in this study are defined as follows:

• **Neutrosophic Logic**: A logical framework that models truth, indeterminacy, and falsity as independent components.

• **Neutrosophic Set**: A generalization of classical and fuzzy sets characterized by truth, indeterminacy, and falsity membership functions.

•**Neutrosphic Real Number:**A number of the form $X = x + yI,\ where\ x, y \in R\ and\ I\ representsindeter$min a cy

• **Neutrosophic Function**: A function whose domain or range consists of neutrosophic real numbers.

• **Neutrosophic Integral**: An extension of the classical integral to neutrosophic-valued functions.

• **Improper Integral**: An integral defined over an infinite interval or involving an unbounded integrand.

• **Improper Neutrosophic Integral**: A neutrosophic integral in which the interval of integration is infinite, or the integrand is not defined at one or more points within the interval.

• **Convergence**: An improper neutrosophic integral is said to be convergent if its defining limit exists and is finite in the neutrosophic sense.

• **Divergence**: An improper neutrosophic integral is divergent if the defining limit does not exist or is infinite.

***1.3 Scope and Limitations of the Study***

This study is limited to single-variable neutrosophic calculus, focusing specifically on improper integrals. The scope includes improper neutrosophic integrals defined over infinite intervals and those involving singularities within finite intervals.

The study does not consider multivariable neutrosophic integrals, numerical approximation methods, or computational simulations. Applications to differential equations, probability theory, and engineering problems are also beyond the scope of this work.

***1.4 Significance of the Study***

This study is significant for several reasons. First, it contributes to the theoretical development of neutrosophic calculus by extending the classical theory of improper integrals into the neutrosophic framework. Second, it provides a clearer understanding of how indeterminacy affects convergence and divergence behavior. Third, the study serves as a useful reference for undergraduate students and researchers interested in modern generalizations of classical analysis. Furthermore, the results of this study may provide a foundation for future research in applied neutrosophic mathematics, particularly in areas where uncertainty and indeterminacy are inherent.

## 2. Historical Review of Classical Integration Theory

The origins of integration theory can be traced to the seventeenth century with the independent discoveries of Isaac Newton and Gottfried Wilhelm Leibniz. Their formulation of differential and integral calculus provided the first systematic tools for solving problems involving areas, volumes, and accumulated quantities. Early integration methods were primarily geometric in nature and were applied to functions defined over finite intervals with well-behaved properties .

As calculus matured during the eighteenth century, mathematicians encountered increasingly complex problems that exposed the limitations of early integration techniques. These challenges motivated the development of more rigorous definitions of the integral. In the nineteenth century, Augustin-Louis Cauchy introduced the use of limits to formalize integration, thereby laying the foundation for modern analysis (see [20]). Cauchy's approach emphasized precision and convergence, which became central themes in integration theory. Bernhard Riemann further advanced integration theory by introducing the Riemann integral, which provided a precise definition of integration based on partitions of intervals and summation processes. Riemann's work clarified the conditions under which functions are integrable and highlighted the importance of convergence, especially when dealing with infinite processes (see [19]). These developments significantly influenced the study of improper integrals. The introduction of measure theory by Henri Lebesgue marked another milestone in the history of integration. Lebesgue integration extended the class of integrable functions and offered greater flexibility than the Riemann approach. Nevertheless, improper integrals retained their relevance, particularly in applications involving infinite domains and singularities. Thus, classical integration theory evolved through successive refinements that emphasized rigor, generality, and convergence.

***2.1Improper Integrals***

The concept of improper integrals arose naturally from the historical evolution of integral calculus as mathematicians sought to extend the applicability of definite integrals beyond finite intervals and bounded functions. In the early development of calculus during the seventeenth and eighteenth centuries, integration was primarily concerned with calculating areas and accumulated quantities for functions that were continuous and well-defined on closed and bounded intervals. However, as mathematical analysis advanced, it became increasingly clear that many important functions of theoretical and practical significance did not satisfy these restrictive conditions. One of the earliest motivations for improper integrals came from problems involving infinite intervals, such as the computation of areas under curves extending indefinitely along the horizontal axis. Classical examples include integrals of exponential and rational functions defined on intervals of the form [a, ∞) or (−∞, ∞). Similarly, mathematicians encountered functions that became unbounded at certain points within an otherwise finite interval, such as $f(x)$ = 1as $x$ , gets $near\ \ x$ = 0. Mathemathecally, this could better be expressed as : $\lim_{x \to 0} f(x) = 1$. These situations could not be handled using the traditional definition of the definite integral, thereby necessitating a broader conceptual framework. Early treatments of such integrals were largely informal and relied on intuitive arguments. However, the lack of rigor led to inconsistencies and paradoxes, particularly, concerning convergence and divergence. A major turning point occurred with the work of Augustin-Louis Cauchy, who introduced a rigorous limit-based approach to integration. Cauchy formalized improper integrals by defining them as limits of proper integrals, thereby embedding them firmly within the emerging framework of mathematical analysis [20]. This approach ensured that the evaluation of improper integrals adhered to precise convergence criteria, eliminating ambiguity. Subsequent developments in the nineteenth century further strengthened the theory of improper integrals. The introduction of the Riemann integral provided a systematic definition of integration based on partitions and sums, while also clarifying the role of limits in defining improper integrals. Later, the Lebesgue integral extended the class of integrable functions even further, although improper integrals continued to be treated primarily through limiting processes within both Riemann and Lebesgue frameworks [19]. From a structural standpoint, improper integrals were classified into two principal categories. Improper integrals of the first kind arise when one or both limits of integration are infinite, whereas improper integrals of the second kind occur when the integrand becomes unbounded at one or more points within the interval of integration. This classification proved essential for the systematic study of convergence, as it enabled mathematicians to develop a variety of analytical tools such as comparison tests, limit comparison tests, and p-tests. These convergence criteria remain fundamental in real analysis and are widely applied in physics, engineering, and probability theory. Despite their analytical strength and broad applicability, classical improper integrals are grounded in a deterministic mathematical framework. They assume that function values, limits, and convergence behavior are precisely defined and free from ambiguity. In many real-world and theoretical contexts, however, information may be incomplete, inconsistent, or indeterminate. Classical improper integrals are not equipped to handle such uncertainty, as they do not incorporate any mechanism for representing indeterminacy within the integrand or the limiting process. These inherent limitations highlighted the need for alternative mathematical frameworks capable of extending integration theory beyond deterministic settings. Over time, this motivation contributed to the development of uncertainty-based theories, including fuzzy calculus and, more recently, neutrosophic calculus. Within this broader historical context, improper neutrosophic integrals can be viewed as a natural and necessary generalization of classical improper integrals, designed to accommodate indeterminacy explicitly while preserving analytical rigor.

### *2.2 Historical Review of Neutrosophic Theory*

Neutrosophic theory was formally introduced by Florentin Smarandache in the year 1998 as a generalization of classical and fuzzy uncertainty models, providing a flexible framework for handling truth, falsity, and indeterminacy independently. The roots of neutrosophy, however, are philosophical, as Smarandache initially explored the theory to examine the interplay between affirmation, negation, and neutrality in human reasoning and philosophical logic. This philosophical foundation emphasized that reality and knowledge often contain elements of contradiction and incompleteness, phenomena that traditional logic and probability could not fully capture.

Following its conceptual inception, neutrosophy evolved into a formal mathematical theory with rigorous applications in logic, set theory, probability, mathematical analysis, etc.(see [13] and [15]). The mathematical formulation introduced the concept of a neutrosophic set, where each element is characterized by three independent components: truth (T), indeterminacy (I), and falsity (F). These components are typically expressed as subsets of the real standard or nonstandard interval ] −0 , 1+ [ , allowing for degrees of membership beyond conventional boundaries. This feature significantly differentiates neutrosophic theory from classical fuzzy or intuitionistic fuzzy frameworks, which enforce normalization constraints such as : $\mu(x) + \nu(x) \leq 1$ , thereby limiting the representation of conflicting or incomplete information. The independence of the three components is a defining hallmark of neutrosophic theory. In practical terms, truth, indeterminacy, and falsity can vary freely without restrictions, reflecting the complex and sometimes contradictory nature of real-world data. This property makes neutrosophic theory particularly suitable for modeling phenomena where information is partial, inconsistent, or uncertain, such as in decision-making systems, artificial intelligence, knowledge representation, and engineering analysis [13]. Over the years, neutrosophic theory has inspired a variety of extensions and applications, including single-valued neutrosophic sets, interval neutrosophic sets, and neutrosophic probability, each adapting the original framework for specific computational or analytical needs. Its versatility has positioned neutrosophic theory as a unifying paradigm, encompassing classical, fuzzy, and intuitionistic fuzzy approaches as special cases, while simultaneously providing enhanced modeling capability for indeterminacy and inconsistency, areas traditionally neglected by earlier uncertainty frameworks.

## 3. Conceptual Review

### *3.1 Neutrosophic Logic*

Neutrosophic logic represents a significant generalization of classical and fuzzy logic by allowing propositions to possess independent degrees of truth (T), indeterminacy (I), and falsity (F) (see [13 and [15]). Unlike classical binary logic, which categorizes statements strictly as true or false, or fuzzy logic, which assigns a degree of membership reflecting vagueness, neutrosophic logic explicitly accommodates uncertainty, incompleteness, and contradiction. This independence of the three components is crucial in modeling complex scenarios, such as conflicting evidence, paradoxical statements, or incomplete knowledge systems, which traditional logical frameworks cannot adequately address. The mathematical formalization of neutrosophic logic provides the foundation for all subsequent neutrosophic constructs, including sets, numbers, functions, and calculus. Its flexibility allows each component to vary freely within extended intervals, reflecting real-world phenomena where truth, falsity, and indeterminacy are not necessarily complementary or mutually exclusive. This logical framework enables quantitative reasoning about indeterminacy, providing a bridge between abstract logic and practical mathematical modeling. Furthermore, neutrosophic logic has found applications in decision-making, artificial intelligence, data analysis, and information theory, where classical or fuzzy logic fails to capture the full spectrum of uncertainty. Its conceptual clarity and generality make it a versatile tool for formalizing problems where uncertainty is multifaceted, serving as the backbone for advanced neutrosophic mathematics.

### *3.2 Neutrosophic Sets and Numbers*

Building upon neutrosophic logic, neutrosophic sets extend the idea of fuzzy sets by assigning three independent membership degrees to each element: truth (T), indeterminacy (I), and falsity (F) [13]. Unlike fuzzy sets, which only quantify the degree of membership, neutrosophic sets allow a more comprehensive representation of uncertainty. For example, an element in a neutrosophic set could be 0.7 true, 0.2 indeterminate, and 0.5 false simultaneously, reflecting ambiguous or conflicting information. Neutrosophic sets serve as a bridge between logical theory and mathematical analysis, providing a structured way to incorporate uncertainty into classical set operations. Union, intersection, and complement operations in neutrosophic sets preserve the independent nature of T, I, and F components, ensuring that indeterminacy is not artificially constrained. This property is particularly valuable in fields such as knowledge representation, expert systems, and decision sciences, where real-world information is rarely precise or complete. Complementing neutrosophic sets, neutrosophic numbers extend real numbers by integrating an indeterminate component. These numbers enable computations in uncertain environments while maintaining algebraic consistency. Vasantha Kandasamy and Smarandache in the year 2011 established arithmetic operations for neutrosophic numbers, ensuring closure under addition, multiplication, and other standard operations, even when the indeterminate component is present. Neutrosophic numbers are essential for propagating uncertainty through calculations, serving as the building blocks for neutrosophic functions, calculus, and integration.

### *3.3 Neutrosophic Functions and Calculus*

Neutrosophic functions are mappings in which the inputs, outputs, or both can be neutrosophic numbers (In other words, each of them can better be expressed as a + Ib, where a and b are real numbers and I, the indeterminacy representation). By preserving the indeterminate components during functional operations, neutrosophic functions enable the modeling of systems where uncertainty is inherent. For instance, a neutrosophic function can describe the behavior of a physical system subject to measurement errors, incomplete information, or ambiguous parameters, ensuring that all components of uncertainty are accounted for in the output. Neutrosophic calculus extends classical calculus to accommodate these indeterminacies. Limits, derivatives, and integrals are generalized to operate on neutrosophic numbers and functions, preserving the independence of truth, indeterminacy, and falsity throughout computations. Smarandache [15] introduced the foundational ideas of neutrosophic calculus, while subsequent researchers formalized differentiation rules, integration techniques, and convergence criteria for neutrosophic functions. This generalized calculus is critical for applications in engineering, physics, and applied mathematics, where traditional deterministic analysis fails to capture the full spectrum of uncertainty. By allowing for precise treatment of indeterminacy, neutrosophic calculus provides the analytical tools needed to study dynamic systems, optimization problems, and mathematical models under incomplete or contradictory information.

### *3.4 Improper Integrals*

Improper integrals arise when the integration process involves unbounded functions or infinite intervals, situations not addressed by standard definite integrals. They are formally defined using limits, and their convergence or divergence is analyzed using classical techniques such as the comparison test, ratio test, and p-test. These integrals are foundational in many areas of applied mathematics, physics, and engineering, serving as the baseline for extending integration into neutrosophic frameworks. By understanding classical improper integrals, researchers can systematically generalize integration to functions and numbers that carry degrees of indeterminacy, enabling computations in environments where uncertainty is intrinsic. This provides a link between rigorous classical analysis and neutrosophic integration, ensuring that the extended framework retains mathematical consistency while incorporating the flexibility required for real-world applications.

## 4.Empirical And Analytical Review of The Improper Neutrosophic Integrals

The study of improper neutrosophic integrals is a relatively recent area of research that builds upon classical improper integrals by incorporating the indeterminacy component inherent in neutrosophic mathematics. Unlike classical improper integrals, which focus solely on convergence or divergence of functions over unbounded intervals or involving unbounded functions, neutrosophic improper integrals introduce an additional layer of complexity. Here, the truth, indeterminacy, and falsity components of a neutrosophic function interact with the integration process, influencing both the value and the convergence behavior of the integral. Analytical research in this domain has been limited but impactful. A significant contribution is the work carried out in the year 2025, which systematically investigated improper neutrosophic integrals of the first and second kinds. Their study provided rigorous definitions for these integrals, generalizing classical concepts by explicitly accounting for indeterminacy. The researchers examined the conditions under which these integrals converge, highlighting the crucial role that the indeterminate component plays in determining convergence criteria. Specifically, they demonstrated that even when the truth and falsity components alone might suggest convergence, the presence of significant indeterminacy can alter the outcome, leading to divergence or indeterminate results. This distinction underscores a key property of neutrosophic integrals: the interplay between certainty and uncertainty components affects the analytical behavior of the integral in ways that classical analysis cannot predict. For instance, in classical improper integrals, convergence is determined solely by the behavior of the function near infinity or near singularities. In contrast, in neutrosophic improper integrals, the distribution of indeterminacy across the domain also contributes to convergence or divergence, making the analysis richer and more nuanced. From an empirical perspective, although numerical simulations and applications of improper neutrosophic integrals are still emerging, preliminary studies indicate their utility in modeling systems with incomplete or ambiguous information

## 5. Methodology

### *5.1 The Improper Neutrosophic Integrals and Their Convergence Properties*

Classical integral calculus provides a rigorous framework for measuring quantities under precise and deterministic assumptions. However, many real-world problems involve indeterminacy, inconsistency, and incomplete information, which cannot be adequately modeled using classical or even fuzzy integrals alone. Neutrosophic analysis, introduced to handle truth, indeterminacy, and falsity simultaneously, extends the classical and fuzzy frameworks by explicitly incorporating indeterminacy into mathematical structures.

### *5.2 The Neutrosophic Numbers and Functions*

#### 5.2.1 Neutrosophic Numbers

A neutrosophic number $N$ is generally expressed in the form

$$N = (T, I, F),$$

where:

- $T$ , represents the degree of truth,
- $I$, represents the degree of indeterminacy,
- $F$, represents the degree of falsity,

with $T, I, F \subseteq [0,1]$, and without the restriction that $T + I + F = 1$.

In practical analysis, single-valued neutrosophic numbers (SVNNs) are commonly used, where $T, I, F \in [0,1]$.

#### 5.2.2 Neutrosophic Functions

A neutrosophic-valued function $f_N$defined on an interval $D \subseteq \mathbb{R}$ could be written as :

$$f_N(x) = \left(T_f(x), I_f(x), F_f(x)\right), x \in D.$$

Each component function $T_f(x), I_f(x), F_f(x)$is assumed to be real-valued and measurable on $D$.

### *5.3 Neutrosophic Definite Integral*

Let $f_N(x) = (T_f(x), I_f(x), F_f(x))$ be a neutrosophic function defined on a closed and bounded interval $[\mathrm{a,b}]$. The neutrosophic definite integral of $f_N$ over $[\mathrm{a,b}]$could be defined componentwiseas :

$$\int_a^b f_N(x)\,dx = \left(\int_a^b T_f(x)\,dx, \int_a^b I_f(x)\,dx, \int_a^b F_f(x)\,dx\right)$$

provided that all the three integrals exist in the Riemann or Lebesgue sense.

This definition ensures compatibility with classical integration theory with the fact that it preserves the neutrosophic structure.

### *5.4 Improper Neutrosophic Integrals*

Improper neutrosophic integrals arise when either the interval of integration is unbounded or the integrand becomes unbounded within the interval.

5.4.1 Improper Integrals over Infinite Intervals

Let $f_N(x) = (T_f(x), I_f(x), F_f(x))$be defined on $[a, \infty)$. The improper neutrosophic integral could be defined as :

$$\int_a^{\infty} f_N(x)\, dx = \lim_{b\to\infty} \int_a^b f_N(x)\, dx,$$

provided that the limit exists componentwise, that is,

$$\lim_{b\to\infty} \int_a^b T_f(x)\, dx, \lim_{b\to\infty} \int_a^b I_f(x)\, dx, and \lim_{b\to\infty} \int_a^b F_f(x)\, dx$$

all exist and are finite (i.e<∞ ).

5.4.2 Improper Integrals with Singularities

If $f_N(x)$is unbounded at a point $c \in (a, b)$, then the improper neutrosophic integral could now, be defined by :

$$\int_a^b f_N(x)\, dx = \lim_{\varepsilon\to 0^+} \left( \int_a^{c-\varepsilon} f_N(x)\, dx + \int_{c+\varepsilon}^b f_N(x)\, dx \right),$$

whenever the limits exist componentwise.

***5.5 Convergence Of Improper Neutrosophic Integrals***

5.5.1 Definition of Convergence

An improper neutrosophic integral

$$\int_a^{\infty} f_N(x)\, dx$$

is said to converge if and only if each of the component improper integrals $\int_a^{\infty} T_f(x)\, dx$, $\int_a^{\infty} I_f(x)\, dx$, and $\int_a^{\infty} F_f(x)\, dx$, converges in the classical sense.

If at least one component diverges, the neutrosophic integral is said to diverge (or simply, not convergent).

5.5.2 The Epsilon – Definition ( ε–definition) and Cauchy Criterion for Improper Neutrosophic Integrals

Let

$$I_N(b) = \int_a^b f_N(x)\, dx \in \mathbb{R}^3,$$

denote the vector of partial integrals of the neutrosophic function $f_N$.

The improper neutrosophic integral

$$\int_a^{\infty} f_N(x)\, dx$$

is said to **converge** if there exists a vector $L_N \in \mathbb{R}^3$such that

$$\lim_{b\to\infty} I_N(b) = L_N.$$

Equivalently, the integral converges if and only if for every $\varepsilon > 0$, there exists $M > a$ (the lower limit) such that for all $b_1, b_2 \geq M$,

$$\| I_N(b_1) - I_N(b_2) \| < \varepsilon,$$

where $\|\cdot\|$denotes any norm on $\mathbb{R}^3$.

This condition is known as the **Cauchy criterion** for improper neutrosophic integrals. Since all norms on finite-dimensional vector spaces are equivalent, convergence is independent of the choice of norm.

***5.6 Fundamental Convergence Theorems***

**Theorem 3.1 (Componentwise Convergence Theorem)**

Let $f_N(x) = \left(T_f(x), I_f(x), F_f(x)\right)$ be a neutrosophic function defined on $[a, \infty)$. If each of the improper integrals

$$\int_a^{\infty} T_f(x)\, dx, \int_a^{\infty} I_f(x)\, dx, and \int_a^{\infty} F_f(x)\, dx$$

converges, then the improper neutrosophic integral $\int_a^{\infty} f_N(x)\, dx$ converges.

**Theorem 3.2 (The Comparison Test for Neutrosophic Integrals)**

Let $f_N(x) = \left(T_f(x), I_f(x), F_f(x)\right)$ and $g_N(x) = \left(T_g(x), I_g(x), F_g(x)\right)$ be neutrosophic functions defined on $[a, \infty)$ such that

$$0 \le T_f(x) \le T_g(x), 0 \le I_f(x) \le I_g(x), 0 \le F_f(x) \le F_g(x)$$

for all sufficiently large $x$. If $\int_a^\infty g_N(x)\, dx$converges, then $\int_a^\infty f_N(x)\, dx$ also converges.

**Theorem 3.3 (The Absolute Convergence Theorem)**

If

$$\int_a^\infty |\, T_f(x) \,|\; dx, \int_a^\infty |\, I_f(x) \,|\; dx, and \int_a^\infty |\, F_f(x) \,|\; dx$$

all converge, then the improper neutrosophic integral $\int_a^\infty f_N(x)\, dx$ converges.

***5.7 Conditional Convergence In Neutrosophic Sense***

An improper neutrosophic integral is said to be conditionally convergent if it converges componentwise, but at least one of the component integrals fails to converge absolutely. This phenomenon mirrors conditional convergence in classical analysis and highlights the sensitivity of neutrosophic integrals to indeterminacy components.

## 6. Results and Applications

This aspect applies the theoretical results developed in the previous section to concrete problems involving improper neutrosophic integrals. Emphasis is placed on explicit computation, convergence verification, and interpretation of truth, indeterminacy, and falsity components. Through carefully chosen examples, the section demonstrates how improper neutrosophic integration extends classical improper integrals while preserving mathematical rigor and offering additional analytical flexibility.

***6.1 Basic Improper Neutrosophic Integrals***

Improper Integrals over Infinite Intervals

Consider a neutrosophic-valued function

$$F(x) = \langle T(x), I(x), F(x) \rangle,$$

where each component function is real-valued and defined on $[a, \infty)$. Here, the entire $F(x)$ is being genetated by :$T(x), I(x), and\;\; F(x)$

The improper neutrosophic integral could also be defined as :

$$\int_a^\infty F(x)\, dx = \langle \int_a^\infty T(x)\, dx, \int_a^\infty I(x)\, dx, \int_a^\infty F(x)\, dx \rangle,$$

provided that all three component integrals converge.

**Example 6.1**

Let

$$F(x) = \left\langle e^{-x}, \frac{1}{1+x^2}, e^{-2x} \right\rangle, x \ge 0.$$

Here,$(T(x), I(x), F(x)) \equiv (e^{-x}, \frac{1}{1+x^2}, e^{-2x})$, and each component is positive and integrable on $[0, \infty)$. Hence,

$$\int_0^\infty F(x)\, dx = \left\langle 1, \frac{\pi}{2}, \frac{1}{2} \right\rangle.$$

Thus, the improper neutrosophic integral converges absolutely.

***6.2 The Improper Integrals with Singularities***

Integrals with Finite Singular Points

Let $F(x) = \langle T(x), I(x), F(x) \rangle$be defined on $(a, b]$, where at least one component has a singularity at $x = a$. The improper neutrosophic integral is defined by

$$\int_a^b F(x)\, dx = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b F(x)\, dx,$$

provided the limit exists in $\mathbb{R}^3$.

**Example 6.2**

Let

$$F(x) = \left\langle x^{-\frac{1}{2}}, x^{-\frac{1}{3}}, x^{-2} \right\rangle, 0 < x \le 1.$$

Then

$$\int_0^1 F(x)\,dx = \left\langle \int_0^1 x^{-\frac{1}{2}}\,dx, \int_0^1 x^{-\frac{1}{3}}\,dx, \int_0^1 x^{-2}\,dx \right\rangle$$

The first two integrals converge, while the third diverges. Therefore, the improper neutrosophic integral diverges by the divergence criterion established.

***6.3 The Oscillatory Improper Neutrosophic Integrals***

Oscillatory behavior often arises in applications involving wave motion and signal analysis. Consider neutrosophic functions whose components include oscillatory terms.

**Example 6.3**

Let

$$F(x) = \left\langle \frac{\sin x}{x}, \frac{\cos x}{x}, \frac{1}{x} \right\rangle, x \geq 1.$$

The first two components converge conditionally, while the third diverges. Hence, the improper neutrosophic integral diverges, despite partial component convergence.

This example highlights the necessity of componentwise convergence for the neutrosophic integrability.

***6.4 Application of Comparison Tests***

Comparison principles provide a practical method for establishing convergence without explicit evaluation.

**Example 6.4**

Let

$$F(x) = \left\langle \frac{1}{x^p}, \frac{1}{x^q}, \frac{1}{x^r} \right\rangle, x \geq 1,$$

where $p, q, r > 0$. The improper neutrosophic integral converges if and only if

$$p > 1, q > 1, r > 1.$$

This follows directly from the classical p-test applied component-wise.

***6.5 The Parameter-Dependent Neutrosophic Integrals***

Consider a family of neutrosophic functions depending on a real parameter $\alpha$:

$$F_\alpha(x) = \left\langle e^{-\alpha x}, \frac{1}{1+\alpha x}, e^{-2\alpha x} \right\rangle$$

The improper neutrosophic integral $\int_0^\infty F_\alpha(x)dx$ converges if and only if $\alpha > 0$.

For $\alpha > 0$, each component converges. If $\alpha \leq 0$, at least one component diverges, implying divergence of the neutrosophic integral.

A simple chart for demonstration :

***6.6 The Regulatory Principles Involved in the New Neutrosophic Integrals and Their Operational Outcomes***

Definition of neutrosophic integral

A neutrosophic integral is an extension of classical integral, handling imprecise, uncertain, or incomplete information using neutrosophic numbers (T,I, F).

| <u>Regulatory Principles</u> | <u>Operational Outcomes</u> |
|---|---|
| Uncertainty Handling: | |
| Additivity: f(f(x)+g(x)) = ff(x) + fg(x) | Neutrosophic integrals capture and quantify |
| Homogeneity: faf(x) =aff(x) | uncertainty, providing a more comprehensive result. |
| Monotonicity: f(x) ≤ g(x) impliesff(x) ≤bfg(x) | |
| Imprecise Data Integration: | |
| Neutrosophic Linearity : f(aT +bl+cF) = afT+bfl+cfF | Handles incomplete or imprecise data,enabling more robust decision-making. |
| Fuzzy and classical results | Reduces to classical results when uncertainty is minimal |

## 7. Conclusion

The neutrosophic integration ensures that the extended mathematical framework retains mathematical consistent while incorporating the flexibility required for real-world applications, taking care of the suffacingexegences, vaqueness, and other forms of uncertainties from time to time.

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