

Iterative Refinement in Neutrosophic Logic: A Unified Framework for n -Refined and Partitioned Neutrosophic Models

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ABSTRACT

A Neutrosophic Set models uncertainty via three memberships—truth, indeterminacy, and falsity. We develop an n -Refined Neutrosophic Logic (n -RNL) that splits these into interpretable subcomponents and define refined quadripartitioned, pentapartitioned, and heptapartitioned families capturing contradiction, ignorance, unknown, and relative truth/falsity. Connectives are given blockwise by a t -norm/ t -conorm pair. We prove merge/split homomorphisms that (i) reduce refined models to classical ones and (ii) embed unrefined semantics into refined spaces. We further introduce *Iterative Refined Neutrosophic Logic* (IRNL), allowing repeated, symmetric refinements with functorial stability of evaluations. Numerical examples from clinical decision support, autonomous driving, and fraud detection illustrate granular reasoning under heterogeneous evidence and priority schemes.

Keywords: Iterative Refined Neutrosophic Logic, Refined quadripartitioned Neutrosophic Logic, Refined pentapartitioned Neutrosophic Logic, Refined heptapartitioned Neutrosophic Logic

1 Introduction

1.1 Neutrosophic Sets and Their Extensions

The mathematical modeling of uncertainty, vagueness, incomplete information, and inconsistent evidence has become an important research topic across pure and applied mathematics. Several theoretical frameworks have been developed for this purpose, including Classical Sets, Fuzzy Sets [1], Intuitionistic Fuzzy Sets, hypersoft sets [2, 3] and Neutrosophic Sets [4]. Among these frameworks, Neutrosophic Set theory provides a particularly flexible representation of uncertain information by independently describing truth, indeterminacy, and falsity.

A Neutrosophic Set assigns to each element three membership degrees: a truth-membership degree T , an indeterminacy-membership degree I , and a falsity-membership degree F . In the standard numerical setting, these values belong to $[0, 1]$ and satisfy

$$0 \leq T + I + F \leq 3.$$

The independence of the three components enables Neutrosophic Sets to represent incomplete, ambiguous, and even mutually inconsistent information more explicitly than many classical uncertainty models [5, 6, 4].

A substantial number of extensions of Neutrosophic Sets have subsequently been proposed. Representative examples include Bipolar Neutrosophic Sets[7, 8], HyperNeutrosophic Sets[9, 10], Hesitant Neutrosophic Sets[11], Double-Valued Neutrosophic Sets[12, 13], and q -Rung Orthopair Neutrosophic Sets [14, 15, 16]. These extensions have attracted considerable attention because they permit different kinds of uncertainty, polarity, hesitation, multiplicity, and structural complexity to be represented within a unified neutrosophic perspective.

Neutrosophic Sets also provide a broad conceptual framework encompassing or extending several earlier uncertainty models, including Fuzzy Sets [1, 17], Intuitionistic Fuzzy Sets [18, 19], Vague Sets[20, 21], Picture Fuzzy Sets [22, 23], Uncertain Sets [24, 25], and Hesitant Fuzzy Sets [26, 27]. These theories have been applied in numerous domains, including control theory, engineering, computer science, applied mathematics, medical analysis, information fusion, and decision-making. A brief comparison of Classical, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets is provided in Table 1.

Despite the expressive power of the basic triplet (T, I, F) , a single indeterminacy component may be insufficient when the source or nature of uncertainty must be explicitly distinguished. For example, uncertainty may arise from contradictory evidence, insufficient information, missing observations, undefined conditions, or context-dependent evaluations. This observation has motivated the development of partitioned neutrosophic models.

Important examples include Quadripartitioned[28, 29], Pentapartitioned[30, 31], and Heptapartitioned Neutrosophic Sets[32, 33]. A Quadripartitioned Neutrosophic Set separates contradiction from ignorance [7, 34]. A Pentapartitioned Neutrosophic Set further distinguishes an unknown component [35, 36]. A Heptapartitioned Neutrosophic Set additionally introduces relative truth and relative



Table 1: A brief comparison of Classical, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets

Model	Membership components	Brief description
Classical Set	$\chi_A(x) \in \{0, 1\}$	Each element x either belongs to A , with $\chi_A(x) = 1$, or does not belong to A , with $\chi_A(x) = 0$. No intermediate degree of membership is permitted.
Fuzzy Set	$\mu_A(x) \in [0, 1]$	Assigns to each element x a membership degree $\mu_A(x)$, thereby representing gradual membership between 0 and 1.
Intuitionistic Fuzzy Set	$\mu_A(x), \nu_A(x) \in [0, 1]$, with $0 \leq \mu_A(x) + \nu_A(x) \leq 1$	Uses both a membership degree $\mu_A(x)$ and a non-membership degree $\nu_A(x)$. The hesitation degree is given by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$.
Neutrosophic Set	$T_A(x), I_A(x), F_A(x) \in [0, 1]$	Assigns independent truth, indeterminacy, and falsity degrees to each element x , subject to $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$, allowing the representation of incomplete and inconsistent information.

falsity, thereby allowing context-dependent evidence to be represented [37, 38]. These models provide increasingly detailed descriptions of the origins and types of uncertainty and have been studied in decision-making and related analytical applications. Their principal features are summarized in Table 2.

Table 2: A concise overview of selected neutrosophic uncertainty models

Model	Membership components	Brief description
Neutrosophic Set	$T(x)$: truth, $I(x)$: indeterminacy, $F(x)$: falsity	Represents the truth, indeterminacy, and falsity associated with each element, subject to $0 \leq T(x) + I(x) + F(x) \leq 3$.
Quadripartitioned Neutrosophic Set	$T(x)$: truth, $C(x)$: contradiction, $G(x)$: ignorance, $F(x)$: falsity	Separates conflicting evidence, represented by C , from a lack of information, represented by G , and satisfies $0 \leq T + C + G + F \leq 4$.
Pentapartitioned Neutrosophic Set	$T(x)$: truth, $C(x)$: contradiction, $G(x)$: ignorance, $U(x)$: unknown, $F(x)$: falsity	Distinguishes contradiction, partial or inconclusive information, and genuinely unknown or undefined information, subject to $0 \leq T + C + G + U + F \leq 5$.
Heptapartitioned Neutrosophic Set	$T(x)$: truth, $M(x)$: relative truth, $C(x)$: contradiction, $U(x)$: unknown, $I(x)$: ignorance, $K(x)$: relative falsity, $F(x)$: falsity	Adds relative truth and relative falsity to the partition of indeterminacy, supporting highly granular and context-sensitive representations of evidence, with $0 \leq T + M + C + U + I + K + F \leq 7$.

1.2 n -Refined Neutrosophic Logic

Another important extension is n -Refined Neutrosophic Logic [39]. Rather than representing truth, indeterminacy, and falsity by a single component each, this framework decomposes them into indexed families of subcomponents. Thus, a refined neutrosophic truth value may take the form

$$(T_1, \dots, T_p \mid I_1, \dots, I_r \mid F_1, \dots, F_s), \quad p + r + s = n.$$

Each component may represent a particular evidence source, semantic type, priority level, expert opinion, sensor, criterion, or contextual condition.

This refinement provides a more detailed representation of heterogeneous evidence than the ordinary triplet (T, I, F) . In particular, different pieces of information contributing to the same general category, such as truth, need not be aggregated prematurely. Instead, they may be retained as distinct subcomponents and combined only when required by the intended logical or decision-making procedure. Consequently, n -refined models are well suited to systems involving multiple sources of evidence, multi-criteria evaluation, hierarchical information, or differentiated uncertainty mechanisms. Related concepts, such as Refined Fuzzy Sets, are also known in the literature. Indeed, refined uncertainty models have been investigated in a wide variety of research papers [22, 40, 41, 42, 43].

Table 3 presents a concise comparison between the ordinary Neutrosophic Set and the n -Refined Neutrosophic Set.

Although partitioned and refined neutrosophic models share the objective of increasing representational granularity, they address different aspects of that objective. Partitioned models distinguish semantically different kinds of uncertainty, such as contradiction, ignorance, and unknown information. Refined models distinguish multiple subcomponents within a given semantic category. Combining these two principles therefore yields a substantially richer framework in which both the type and internal composition of each uncertainty component can be explicitly represented.



Table 3: A brief comparison between Neutrosophic and n -Refined Neutrosophic Sets

Aspect	Neutrosophic Set	n -Refined Neutrosophic Set
Components	A single triplet (T, I, F) .	Indexed families of refined truth, indeterminacy, and falsity components, such as T_j , I_k , and F_ℓ .
Representation of an element	One vector in $[0, 1]^3$.	A higher-dimensional vector in $[0, 1]^n$.
Granularity	Provides an aggregated representation of uncertainty.	Provides a source-sensitive, type-sensitive, or priority-sensitive representation.
Typical use	General modeling of truth, indeterminacy, and falsity.	Detailed reasoning involving heterogeneous or multilayered evidence.

1.3 Motivation and Research Gap

Many real-world reasoning problems involve evidence that is simultaneously heterogeneous, contradictory, incomplete, context-dependent, and hierarchically structured. For example, a medical decision may depend on laboratory measurements, imaging findings, physician assessments, missing tests, and competing diagnoses. An autonomous system may receive partially conflicting information from cameras, radar, maps, and external communications. A fraud-detection system may combine transaction patterns, identity information, reputation scores, and uncertain behavioral signals. In such situations, the basic neutrosophic triplet may be too coarse because it aggregates structurally different sources of evidence into only three values.

Partitioned neutrosophic models improve semantic granularity by separating different forms of indeterminacy. However, each resulting category is still typically represented by a single scalar. Conversely, n -refined neutrosophic models divide truth, indeterminacy, and falsity into multiple subcomponents, but they do not necessarily distinguish contradiction, ignorance, unknown information, or relative truth and falsity as separate semantic categories.

Accordingly, there remains a need for a unified logical framework that combines semantic partitioning with componentwise refinement. There is also a need to study refinement as a repeatable process rather than as a single one-step decomposition. A component obtained at one level may itself contain several distinguishable sources or subtypes of evidence and may therefore require further refinement. This naturally leads to an iterative architecture in which refinement can be repeatedly applied while preserving the logical structure and compatibility of evaluations across levels.

1.4 Main Contributions

Motivated by the preceding observations, this paper develops a unified framework for refined and iteratively refined neutrosophic logic. The main contributions are summarized as follows.

First, we formulate an n -Refined Neutrosophic Logic (n -RNL) in which truth, indeterminacy, and falsity are represented by finite blocks of indexed components. Logical connectives are defined blockwise and componentwise by means of a chosen t -norm and its dual t -conorm.

Second, we introduce refined versions of Quadripartitioned, Pentapartitioned, and Heptapartitioned Neutrosophic Logic. These models simultaneously distinguish the semantic categories of uncertainty and resolve each category into multiple subcomponents. In particular, the proposed framework includes refined truth, contradiction, ignorance, unknown, relative-truth, relative-falsity, and falsity blocks, depending on the partitioned model under consideration.

Third, we establish canonical merge and split mappings between refined partitioned models and their corresponding less-refined or unrefined counterparts. These mappings clarify how classical neutrosophic structures can be recovered from the proposed models and how lower-dimensional evaluations can be embedded into refined spaces. We also investigate the compatibility of these mappings with the logical operations.

Fourth, we introduce *Iterative Refined Neutrosophic Logic* (IRNL). In this framework, a component may be repeatedly refined into successively more detailed subcomponents, producing a hierarchy of neutrosophic truth-value spaces. This construction supports multilevel modeling while preserving the structural stability of logical evaluations under refinement, merging, and splitting.

Fifth, we present theoretical properties and numerical examples illustrating the proposed framework. The examples demonstrate how refined and iterative neutrosophic evaluations may be used to represent heterogeneous evidence in clinical decision support, autonomous driving, cyber-security, fraud detection, and related decision-making contexts.

Table 4 summarizes the principal refined neutrosophic models considered in this paper.

The relationship among ordinary, refined, and iteratively refined neutrosophic models is summarized in Table 5.

The proposed approach therefore provides a systematic mathematical foundation for modeling uncertainty at multiple semantic and structural levels. It retains the interpretability of partitioned neutrosophic components while introducing the flexibility needed to represent multiple evidence sources and repeated decompositions.

1.5 Organization of the Paper

The remainder of this paper is organized as follows. Section 2 reviews the preliminary concepts and notation used throughout the paper, including Neutrosophic Sets, n -Refined Neutrosophic Logic, and Quadripartitioned, Pentapartitioned, and Heptapartitioned Neutrosophic Sets. Section 3 presents the principal results on Refined Quadripartitioned, Refined Pentapartitioned, and Refined Heptapartitioned Neutrosophic Logics, together with their merge and split mappings, fundamental properties, illustrative applications, and



Table 4: A concise overview of the refined neutrosophic models considered in this paper

Model	Membership components	Brief description
n -Refined Neutrosophic Set	$\{T_\alpha(x)\}$: refined truth, $\{I_\beta(x)\}$: refined indeterminacy, $\{F_\gamma(x)\}$: refined falsity	Decomposes truth, indeterminacy, and falsity into finite indexed families representing distinct sources, types, or priority levels. The ordinary Neutrosophic Set is recovered when each family contains one component.
Refined Quadripartitioned Neutrosophic Set	$\{T_\alpha(x)\}$: refined truth, $\{C_\beta(x)\}$: refined contradiction, $\{G_\gamma(x)\}$: refined ignorance, $\{F_\delta(x)\}$: refined falsity	Refines truth, contradiction, ignorance, and falsity into separate indexed families. It reduces to the ordinary quadripartitioned model when each family is a singleton.
Refined Pentapartitioned Neutrosophic Set	$\{T_\alpha(x)\}$: refined truth, $\{C_\beta(x)\}$: refined contradiction, $\{G_\gamma(x)\}$: refined ignorance, $\{U_\delta(x)\}$: refined unknown, $\{F_\varepsilon(x)\}$: refined falsity	Provides componentwise refinements of truth, contradiction, ignorance, unknown information, and falsity, allowing several uncertainty modes to be tracked simultaneously.
Refined Heptapartitioned Neutrosophic Set	$\{T_\alpha(x)\}$: refined truth, $\{M_\beta(x)\}$: refined relative truth, $\{C_\gamma(x)\}$: refined contradiction, $\{U_\delta(x)\}$: refined unknown, $\{I_\eta(x)\}$: refined ignorance, $\{K_\theta(x)\}$: refined relative falsity, $\{F_\zeta(x)\}$: refined falsity	Refines all seven heptapartitioned components and thereby supports highly granular, context-sensitive modeling of complex and heterogeneous evidence.

Table 5: A brief comparison of ordinary, refined, and iteratively refined neutrosophic models

Model	Components	Brief description
Neutrosophic Set	$T(x), I(x), F(x)$	Assigns to each element a single truth degree, indeterminacy degree, and falsity degree, with $(T(x), I(x), F(x)) \in [0, 1]^3$.
Refined Neutrosophic Set	$\{T_\alpha(x)\}, \{I_\beta(x)\}, \{F_\gamma(x)\}$	Decomposes each of T, I , and F into finitely many indexed components representing heterogeneous evidence.
Iteratively Refined Neutrosophic Set	$\{T_\alpha^{(k)}(x)\}, \{I_\beta^{(k)}(x)\}, \{F_\gamma^{(k)}(x)\}$, for $k \geq 0$	Applies refinement repeatedly, producing a hierarchy of component spaces across refinement levels k . Each level provides a more detailed representation while remaining related to the preceding level through canonical structural mappings.

Iterative Refined Neutrosophic Logic. Section 4 extends the refinement framework to Uncertain Sets by introducing Refined Uncertain Sets, finite-depth and infinitely iterated Refined Uncertain Sets, Multidimensional Iterated Refined Uncertain Sets, and their infinite-dimensional-level counterparts. Finally, Section 5 summarizes the main findings and outlines directions for future research.

2 Preliminaries

This section gathers the basic notions and notation used throughout the paper. Unless explicitly stated otherwise, we work in the finite setting. Recall that the empty set is a subset of every set.

2.1 Neutrosophic Set

The definition of the Neutrosophic Set is presented below.

Definition 2.1 (Neutrosophic Set). [4] Let X be a non-empty set. A *Neutrosophic Set (NS)* A on X is characterized by three membership functions:

$$T_A : X \rightarrow [0, 1], \quad I_A : X \rightarrow [0, 1], \quad F_A : X \rightarrow [0, 1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$



Example 2.2 (Neutrosophic Set in Practice: Spam Email Identification). (cf.[44])

Let X denote the set of incoming emails in a user's inbox. We consider a neutrosophic set A on X , where

$x \in A$ means “email x is spam to certain truth, indeterminacy, and falsity degrees.”

For a particular email $e \in X$, suppose that the following observable signals are available:

- the content-based spam score

$$s_{\text{cont}}(e) = 0.74,$$

where a larger value indicates stronger spam-like characteristics;

- the sender-reputation badness score

$$s_{\text{rep}}(e) = 0.65;$$

- the missing or uncertain feature score

$$m(e) = 0.22,$$

which may arise from partial email headers, unavailable authentication information, or missing DKIM/DMARC results;

- the whitelisting score and conversational-context score

$$w(e) = 0.20, \quad h(e) = 0.15,$$

which provide evidence that the email may be legitimate rather than spam.

We use the probabilistic t -conorm

$$s(a, b) = a + b - ab = 1 - (1 - a)(1 - b)$$

to aggregate independent supporting signals.

Truth membership. The truth membership represents the combined evidence supporting the classification of e as spam. It is defined by

$$\begin{aligned} T_A(e) &= s(s_{\text{cont}}(e), s_{\text{rep}}(e)) \\ &= 1 - (1 - s_{\text{cont}}(e))(1 - s_{\text{rep}}(e)) \\ &= 1 - (1 - 0.74)(1 - 0.65) \\ &= 1 - (0.26)(0.35) \\ &= 0.909. \end{aligned}$$

Indeterminacy membership. The indeterminacy membership represents uncertainty caused by incomplete, unavailable, or unreliable technical information. We set

$$I_A(e) = m(e) = 0.22.$$

Falsity membership. The falsity membership represents the evidence supporting the interpretation that e is a legitimate, non-spam email. It is defined by

$$\begin{aligned} F_A(e) &= s(w(e), h(e)) \\ &= w(e) + h(e) - w(e)h(e) \\ &= 0.20 + 0.15 - (0.20)(0.15) \\ &= 0.32. \end{aligned}$$

Figure 1 summarizes the construction of the three neutrosophic membership values.

Consequently, the neutrosophic assessment of email e is

$$(T_A(e), I_A(e), F_A(e)) = (0.909, 0.22, 0.32).$$

The standard neutrosophic sum condition is satisfied because

$$0 \leq T_A(e) + I_A(e) + F_A(e) = 0.909 + 0.22 + 0.32 = 1.449 \leq 3.$$

Thus, the relatively large truth membership $T_A(e) = 0.909$ indicates strong evidence that the email is spam, primarily because of its content and the sender's poor reputation. The indeterminacy membership $I_A(e) = 0.22$ records the remaining uncertainty caused by incomplete technical information. Meanwhile, the falsity membership $F_A(e) = 0.32$ reflects moderate evidence that the email may be legitimate, arising from whitelisting and its conversational context.

Importantly, these three values are not required to sum to 1. The simultaneous presence of a high truth membership and a nonzero falsity membership allows the model to retain supporting and opposing evidence without prematurely forcing them into a single probability distribution.

Example 2.3 (Neutrosophic Set in Practice: Bacterial Pneumonia Triage). (cf.[45]) Let X be the set of adult patients in an emergency department. Define the neutrosophic set $A \subseteq X$ with meaning

$x \in A$ “to the degree that patient x has bacterial pneumonia.”

For a given patient $p \in X$, suppose we observe:

- Biomarker support (e.g., CRP/procalcitonin) $s_{\text{lab}}(p) = 0.68$,
- Imaging support (chest radiograph) $s_{\text{img}}(p) = 0.55$,



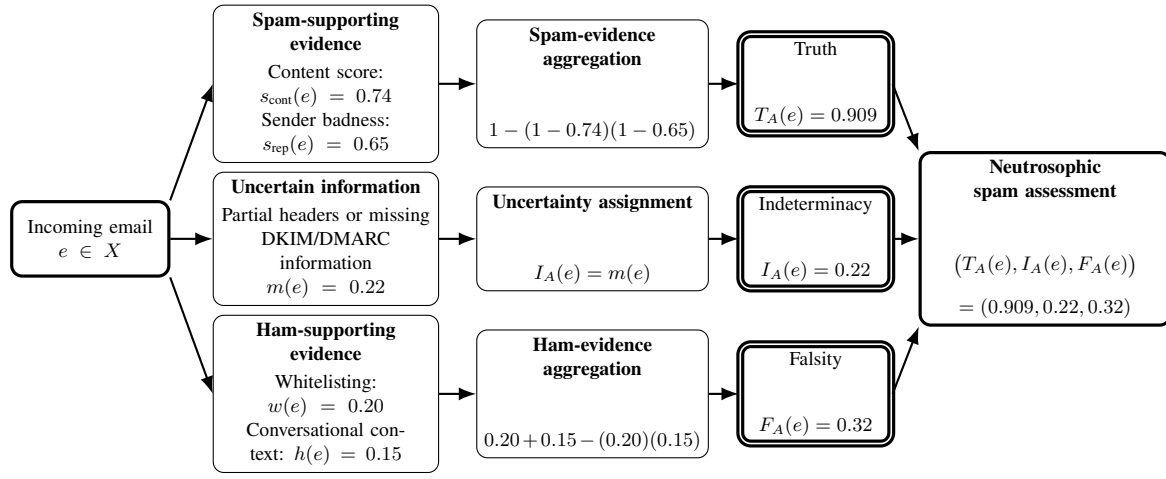


Figure 1: Overview of the neutrosophic spam-email assessment. Spam-supporting evidence determines the truth membership, missing technical information determines the indeterminacy membership, and legitimate-email cues determine the falsity membership.

- Pending culture/diagnostics fraction $u_{\text{pend}}(p) = 0.30$ and incomplete history $u_{\text{hist}}(p) = 0.10$,
- Counter-evidence: viral PCR positivity $f_{\text{viral}}(p) = 0.35$ and alternative diagnosis likelihood $f_{\text{alt}}(p) = 0.20$.

Define the memberships (all in $[0, 1]$) by:

$$\text{Truth (supporting evidence): } T_A(p) = s_{\text{lab}}(p) + s_{\text{img}}(p) - s_{\text{lab}}(p) s_{\text{img}}(p) = 0.68 + 0.55 - 0.68 \cdot 0.55 = 0.856.$$

$$\text{Indeterminacy (unknowns/missingness): } I_A(p) = u_{\text{pend}}(p) + u_{\text{hist}}(p) - u_{\text{pend}}(p) u_{\text{hist}}(p) = 0.30 + 0.10 - 0.30 \cdot 0.10 = 0.37.$$

$$\text{Falsity (counter-evidence): } F_A(p) = f_{\text{viral}}(p) + f_{\text{alt}}(p) - f_{\text{viral}}(p) f_{\text{alt}}(p) = 0.35 + 0.20 - 0.35 \cdot 0.20 = 0.48.$$

Hence

$$(T_A(p), I_A(p), F_A(p)) = (0.856, 0.37, 0.48),$$

and the bound is satisfied:

$$0.856 + 0.37 + 0.48 = 1.706 \leq 3.$$

Concordant lab and imaging provide substantial truth; pending tests and incomplete history contribute to indeterminacy; viral findings and a plausible alternative diagnosis add falsity.

2.2 n-Refined Neutrosophic Logic

n -Refined Neutrosophic Logic splits truth, indeterminacy, and falsity into multiple subcomponents, enabling granular reasoning under heterogeneous evidence with priorities assigned [46].

Definition 2.4 (n -Refined Neutrosophic Logic (n-RNL)). [39, 47] Fix integers $p, r, s \geq 1$ and set $n := p + r + s$. An n -refined neutrosophic truth value is a vector

$$\mathbf{v} = (T_1, \dots, T_p \mid I_1, \dots, I_r \mid F_1, \dots, F_s), \quad T_j, I_k, F_\ell \in [0, 1],$$

where the T -components encode refined kinds of truth, the I -components encode refined kinds of indeterminacy, and the F -components encode refined kinds of falsity; the total dimension is $n = p + r + s$. In the standard (numerical) setting one may assume

$$0 \leq \sum_{j=1}^p T_j + \sum_{k=1}^r I_k + \sum_{\ell=1}^s F_\ell \leq n,$$

while in philosophical/nonstandard variants each component may range in the nonstandard unit interval, with the same n -dimensional refinement principle. The splitting of (T, I, F) into $(T_1, \dots, T_p)$, $(I_1, \dots, I_r)$, $(F_1, \dots, F_s)$ (with $p + r + s = n$) is the hallmark of n -refinement.

A (propositional) n -refined neutrosophic valuation on a language $\mathcal{L}$ is a map

$$\text{Val} : \text{Form}(\mathcal{L}) \longrightarrow [0, 1]^n, \quad \phi \longmapsto \text{Val}(\phi) = (\mathbf{T}(\phi) \mid \mathbf{I}(\phi) \mid \mathbf{F}(\phi)),$$

equipped with the following connectives defined componentwise from a chosen fuzzy t -norm $t : [0, 1]^2 \rightarrow [0, 1]$ and its dual t -conorm $s : [0, 1]^2 \rightarrow [0, 1]$ (both associative, commutative, and monotone).

Conjunction (n-norm). The neutrosophic conjunction $\phi \wedge_n \psi$ is

$$\begin{aligned} \mathbf{T}(\phi \wedge_n \psi) &= t\text{-wise combine } (T_1(\phi), \dots, T_p(\phi)) \text{ with } (T_1(\psi), \dots, T_p(\psi)), \\ \mathbf{I}(\phi \wedge_n \psi) &= s\text{-wise combine } (I_1(\phi), \dots, I_r(\phi)) \text{ with } (I_1(\psi), \dots, I_r(\psi)), \\ \mathbf{F}(\phi \wedge_n \psi) &= s\text{-wise combine } (F_1(\phi), \dots, F_s(\phi)) \text{ with } (F_1(\psi), \dots, F_s(\psi)). \end{aligned}$$



Equivalently, T uses a t -norm, while both I and F use a t -conorm.

Disjunction (n-conorm). The neutrosophic disjunction $\phi \vee_n \psi$ is

$$\begin{aligned}\mathbf{T}(\phi \vee_n \psi) &= s\text{-wise combine the } T\text{-blocks,} \\ \mathbf{I}(\phi \vee_n \psi) &= t\text{-wise combine the } I\text{-blocks,} \\ \mathbf{F}(\phi \vee_n \psi) &= t\text{-wise combine the } F\text{-blocks.}\end{aligned}$$

Equivalently, T uses a t -conorm, while both I and F use a t -norm.

Negation. A basic (involutive) neutrosophic negation swaps truth and falsity blocks and leaves indeterminacy as is:

$$\neg_n \phi : (\mathbf{T}(\phi) \mid \mathbf{I}(\phi) \mid \mathbf{F}(\phi)) \longmapsto (\mathbf{F}(\phi) \mid \mathbf{I}(\phi) \mid \mathbf{T}(\phi)),$$

with optional numeric complementation $x \mapsto 1 - x$ applied componentwise when a strong negation is desired.

Priority (pessimistic/optimistic) variants. When needed, *priority* orderings among the refined subcomponents (e.g. $T < I < F$ for lower-bound n -norm, or $T > I > F$ for upper-bound n -conorm) specialize the above mixers; in the unrefined triplet case this yields the familiar closed forms for $\wedge_n$ and $\vee_n$.

Example 2.5 (Clinical Decision Support: Sepsis Triage (n-RNL with $p = r = s = 2$)). (cf.[48, 49]) We assess two propositions on the same patient:

$$\phi := \text{“the patient has bacterial sepsis”}, \quad \psi := \text{“broad-spectrum antibiotics are indicated”}.$$

We adopt $p = r = s = 2$ (so $n = 6$) with components interpreted as:

$$\begin{aligned}\mathbf{T} &= (T_1, T_2) = (\text{lab/imaging truth, guideline/model truth}), \\ \mathbf{I} &= (I_1, I_2) = (\text{missing-data uncertainty, ambiguity}), \\ \mathbf{F} &= (F_1, F_2) = (\text{counter-evidence, alternative diagnosis}).\end{aligned}$$

Choose the Gödel t -norm and t -conorm: $t(a, b) = \min\{a, b\}$ and $s(a, b) = \max\{a, b\}$. Suppose the valuations are

$$\begin{aligned}\text{Val}(\phi) &= ((0.82, 0.68) \mid (0.40, 0.25) \mid (0.35, 0.20)), \\ \text{Val}(\psi) &= ((0.75, 0.72) \mid (0.30, 0.35) \mid (0.25, 0.40)).\end{aligned}$$

Conjunction $\phi \wedge_n \psi$ (truth via t , indeterminacy/falsity via s):

$$\begin{aligned}\mathbf{T} &= (\min\{0.82, 0.75\}, \min\{0.68, 0.72\}) = (0.75, 0.68), \\ \mathbf{I} &= (\max\{0.40, 0.30\}, \max\{0.25, 0.35\}) = (0.40, 0.35), \\ \mathbf{F} &= (\max\{0.35, 0.25\}, \max\{0.20, 0.40\}) = (0.35, 0.40).\end{aligned}$$

Disjunction $\phi \vee_n \psi$ (truth via s , indeterminacy/falsity via t):

$$\begin{aligned}\mathbf{T} &= (\max\{0.82, 0.75\}, \max\{0.68, 0.72\}) = (0.82, 0.72), \quad \mathbf{I} = (\min\{0.40, 0.30\}, \min\{0.25, 0.35\}) = (0.30, 0.25), \\ \mathbf{F} &= (\min\{0.35, 0.25\}, \min\{0.20, 0.40\}) = (0.25, 0.20).\end{aligned}$$

Negation $\neg_n \phi$ (swap truth/falsity, keep indeterminacy):

$$\text{Val}(\neg_n \phi) = ((0.35, 0.20) \mid (0.40, 0.25) \mid (0.82, 0.68)).$$

The conjunction preserves cautious truth (0.75, 0.68) while exposing combined ambiguity (0.40, 0.35) and accumulated counter-evidence (0.35, 0.40). The disjunction reflects that at least one of $\{\phi, \psi\}$ has strong support while overall uncertainty and opposition drop under the t -norm.

Example 2.6 (Autonomous Driving: Highway Merge Decision (n-RNL with $p = r = s = 2$)). Consider

$$\phi := \text{“the current gap is safe to merge”}, \quad \psi := \text{“the adjacent driver will yield”}.$$

Use $p = r = s = 2$ with

$$\begin{aligned}\mathbf{T} &= (T_1, T_2) = (\text{multi-sensor consensus, V2V/traffic-rule inference}), \\ \mathbf{I} &= (I_1, I_2) = (\text{occlusion, adverse weather}), \\ \mathbf{F} &= (F_1, F_2) = (\text{contrary sensor cues, rule-violation risk}).\end{aligned}$$

Choose the product t -norm $t(a, b) = ab$ and the probabilistic t -conorm $s(a, b) = a + b - ab$. Suppose

$$\begin{aligned}\text{Val}(\phi) &= ((0.88, 0.70) \mid (0.20, 0.35) \mid (0.15, 0.25)), \\ \text{Val}(\psi) &= ((0.60, 0.55) \mid (0.30, 0.25) \mid (0.40, 0.30)).\end{aligned}$$

Conjunction $\phi \wedge_n \psi$ (truth via t , indeterminacy/falsity via s):

$$\mathbf{T} = (0.88 \cdot 0.60, 0.70 \cdot 0.55) = (0.528, 0.385),$$

$$\mathbf{I} = (0.20 + 0.30 - 0.20 \cdot 0.30, 0.35 + 0.25 - 0.35 \cdot 0.25) = (0.44, 0.5125),$$



$$\mathbf{F} = (0.15 + 0.40 - 0.15 \cdot 0.40, 0.25 + 0.30 - 0.25 \cdot 0.30) = (0.49, 0.475).$$

Disjunction $\phi \vee_n \psi$ (truth via s , indeterminacy/falsity via t):

$$\mathbf{T} = (0.88 + 0.60 - 0.88 \cdot 0.60, 0.70 + 0.55 - 0.70 \cdot 0.55) = (0.952, 0.865),$$

$$\mathbf{I} = (0.20 \cdot 0.30, 0.35 \cdot 0.25) = (0.06, 0.0875),$$

$$\mathbf{F} = (0.15 \cdot 0.40, 0.25 \cdot 0.30) = (0.06, 0.075).$$

Negation $\neg_n \phi$:

$$\text{Val}(\neg_n \phi) = ((0.15, 0.25) \mid (0.20, 0.35) \mid (0.88, 0.70)).$$

The conjunction demands joint support, thus lowering truth by multiplication and accumulating uncertainty/opposition via s . The disjunction captures the *any-sufficient* nature of acting safely: high fused truth (0.952, 0.865) with suppressed uncertainty and opposition under t .

2.3 Quadripartitioned, Pentapartitioned, and Heptapartitioned Neutrosophic Set

A quadripartitioned neutrosophic set assigns to each element four independent degrees—truth, contradiction, ignorance, and falsity—each valued in $[0, 1]$ with the total bounded above by 4. This cleanly separates “conflicting evidence” (contradiction) from “insufficient information” (ignorance), enabling a finer representation of uncertainty than a single truth grade [50, 51, 52, 53]. A pentapartitioned neutrosophic set describes each element by five memberships—truth, contradiction, ignorance, unknown, and falsity—again taking values in $[0, 1]$ with the sum at most 5. The explicit “unknown” component distinguishes genuinely missing or undefined observations from “ignorance,” which captures partial yet inconclusive information [36, 54]. A heptapartitioned neutrosophic set equips each element with seven degrees—truth, relative truth, contradiction, unknown, ignorance, relative falsity, and falsity—each in $[0, 1]$ and jointly bounded by 7. The “relative” truth and falsity components encode context-dependent tendencies, supporting highly nuanced reasoning in complex, ambiguous settings [55, 56, 57].

Definition 2.7 (Quadripartitioned Neutrosophic Set). [50, 51, 52, 53] Let U be a non-empty universe. A *quadripartitioned neutrosophic set* (Q-NS) Q on U is defined by

$$Q = \{ \langle x, T_Q(x), C_Q(x), G_Q(x), F_Q(x) \rangle \mid x \in U \},$$

where

$$T_Q, C_Q, G_Q, F_Q : U \longrightarrow [0, 1]$$

are the *truth*, *contradiction*, *ignorance*, and *falsity* membership functions, respectively, satisfying

$$0 \leq T_Q(x) + C_Q(x) + G_Q(x) + F_Q(x) \leq 4 \quad \text{for all } x \in U.$$

Example 2.8 (Quadripartitioned Neutrosophic Set: Evacuation Need in Disaster Response). Let U be the set of households in district D . Consider the Q-NS Q whose intended meaning is

$$x \in Q \text{ “to the degree that household } x \text{ needs immediate evacuation.”}$$

For a specific household $h_{17} \in U$, suppose we have the following measurable indicators:

- Hazard index (ground motion + landslide risk): $H(h_{17}) = 0.80$,
- Structural damage index (rapid visual assessment): $D(h_{17}) = 0.60$,
- Two decision sources: municipal order $S_1(h_{17}) = 1.00$ (“evacuate”) and field officer $S_2(h_{17}) = 0.60$ (“remain”),
- Data completeness (sensors, reports): $\text{Comp}(h_{17}) = 0.75$,
- Stability/benign-evidence index (nearby shelter capacity, route safety): $B(h_{17}) = 0.20$.

Define the Q-NS memberships (all in $[0, 1]$) by

$$T_Q(h_{17}) = \max\{H(h_{17}), D(h_{17})\} = \max\{0.80, 0.60\} = 0.80,$$

$$C_Q(h_{17}) = |S_1(h_{17}) - S_2(h_{17})| = |1.00 - 0.60| = 0.40,$$

$$G_Q(h_{17}) = 1 - \text{Comp}(h_{17}) = 1 - 0.75 = 0.25,$$

$$F_Q(h_{17}) = B(h_{17}) = 0.20.$$

Hence the quadripartitioned tuple is

$$(T_Q(h_{17}), C_Q(h_{17}), G_Q(h_{17}), F_Q(h_{17})) = (0.80, 0.40, 0.25, 0.20),$$

and the total is $0.80 + 0.40 + 0.25 + 0.20 = 1.65 \leq 4$.

Strong hazard/damage support evacuation ($T = 0.80$); there is moderate contradiction between sources ($C = 0.40$) and some ignorance from partial data ($G = 0.25$); a small amount of counter-evidence ($F = 0.20$) comes from local stability factors.

Definition 2.9 (Pentapartitioned Neutrosophic Set). [36, 54] Let U be a non-empty universe. A *pentapartitioned neutrosophic set* (P-NS) P on U is defined by

$$P = \{ \langle x, T_P(x), C_P(x), G_P(x), U_P(x), F_P(x) \rangle \mid x \in U \},$$

where

$$T_P, C_P, G_P, U_P, F_P : U \longrightarrow [0, 1]$$

are the *truth*, *contradiction*, *ignorance*, *unknown*, and *falsity* membership functions, respectively, satisfying

$$0 \leq T_P(x) + C_P(x) + G_P(x) + U_P(x) + F_P(x) \leq 5 \quad \text{for all } x \in U.$$



Example 2.10 (Pentapartitioned Neutrosophic Set: Urgent Antibiotic Decision). (cf.[58]) Let U be the set of patients in an emergency department. Define the P-NS P with meaning

$x \in P$ “to the degree that patient x requires broad-spectrum antibiotics now.”

For a patient $p^* \in U$, suppose:

- Blood culture signal $B(p^*) = 0.82$, procalcitonin elevation $PCT(p^*) = 0.78$,
- Model A (sepsis triage) score $M_A(p^*) = 0.85$, stewardship rule (“withhold”) score $M_S(p^*) = 0.40$,
- Laboratory/missingness ratio $Miss(p^*) = 0.10$,
- Unknowns from pending cultures $Pend(p^*) = 0.30$,
- Viral panel evidence (against bacterial cause) $V(p^*) = 0.25$.

Define the P-NS memberships (all in $[0, 1]$) by

$$\begin{aligned} T_P(p^*) &= \min\{B(p^*), PCT(p^*)\} = \min\{0.82, 0.78\} = 0.78, \\ C_P(p^*) &= |M_A(p^*) - M_S(p^*)| = |0.85 - 0.40| = 0.45, \\ G_P(p^*) &= Miss(p^*) = 0.10, \quad U_P(p^*) = Pend(p^*) = 0.30, \\ F_P(p^*) &= V(p^*) = 0.25. \end{aligned}$$

Thus

$$(T_P, C_P, G_P, U_P, F_P)(p^*) = (0.78, 0.45, 0.10, 0.30, 0.25),$$

with total $0.78 + 0.45 + 0.10 + 0.30 + 0.25 = 1.88 \leq 5$.

Convergent biomarkers support treatment ($T = 0.78$) while a stewardship rule conflicts with the triage model ($C = 0.45$); some evidence is still unknown (pending) and a modest falsity signal arises from viral findings.

Definition 2.11 (Heptapartitioned Neutrosophic Set). [55, 56, 57] Let U be a non-empty universe. A *heptapartitioned neutrosophic set* (H-NS) H on U is defined by

$$H = \{ \langle x, T_H(x), M_H(x), C_H(x), U_H(x), I_H(x), K_H(x), F_H(x) \rangle \mid x \in U \},$$

where

$$T_H, M_H, C_H, U_H, I_H, K_H, F_H : U \longrightarrow [0, 1]$$

are the *truth, relative truth, contradiction, unknown, ignorance, relative falsity, and falsity* membership functions, respectively, satisfying

$$0 \leq T_H(x) + M_H(x) + C_H(x) + U_H(x) + I_H(x) + K_H(x) + F_H(x) \leq 7 \quad \text{for all } x \in U.$$

Example 2.12 (Heptapartitioned Neutrosophic Set: Authenticity of an Online Listing). (cf.[59]) Let U be the set of listings on a marketplace. Consider the H-NS H meaning

$x \in H$ “to the degree that listing x is genuine (not fraudulent).”

For a listing $\ell \in U$, suppose we observe:

- Objective truth (IMEI/serial check passed) $T_{\text{obj}}(\ell) = 0.90$,
- Relative/context truth: seller reputation $R(\ell) = 0.80$, description-photo consistency $M_{\text{cons}}(\ell) = 0.70$,
- Contradiction (metadata mismatches) $C_{\text{mis}}(\ell) = 0.20$,
- Unknown (missing invoice/warranty) $U_{\text{miss}}(\ell) = 0.40$,
- Ignorance (low-resolution photos/incomplete specs) $I_{\text{inc}}(\ell) = 0.30$,
- Relative falsity (price anomaly score) $K_{\text{anom}}(\ell) = 0.50$,
- Objective falsity (hit in theft/blacklist DB) $F_{\text{obj}}(\ell) = 0.10$.

Set the seven memberships by

$$\begin{aligned} T_H(\ell) &= T_{\text{obj}}(\ell) = 0.90, \quad M_H(\ell) = \min\{R(\ell), M_{\text{cons}}(\ell)\} = \min\{0.80, 0.70\} = 0.70, \\ C_H(\ell) &= C_{\text{mis}}(\ell) = 0.20, \quad U_H(\ell) = U_{\text{miss}}(\ell) = 0.40, \quad I_H(\ell) = I_{\text{inc}}(\ell) = 0.30, \\ K_H(\ell) &= K_{\text{anom}}(\ell) = 0.50, \quad F_H(\ell) = F_{\text{obj}}(\ell) = 0.10. \end{aligned}$$

Hence

$$(T_H, M_H, C_H, U_H, I_H, K_H, F_H)(\ell) = (0.90, 0.70, 0.20, 0.40, 0.30, 0.50, 0.10),$$

and the sum is $0.90 + 0.70 + 0.20 + 0.40 + 0.30 + 0.50 + 0.10 = 3.10 \leq 7$.

Strong objective truth (device check) and contextual support (reputation/consistency) coexist with moderate unknown/ignorance and a sizable relative-falsity signal due to price anomaly; objective falsity remains low.

3 Main Results

In this section, we present the main results of this paper.



3.1 Refined Quadripartitioned Neutrosophic Logic

Refined Quadripartitioned Neutrosophic Logic splits truth, contradiction, ignorance, falsity into vector components, combining with t -norms and t -conorms to model uncertainty.

Definition 3.1 (n-Refined Quadripartitioned Neutrosophic Logic (n-RQNL)). Fix integers $p, c, g, f \geq 1$ and put $n := p + c + g + f$. An n -refined quadripartitioned truth value is

$$\mathbf{v} = (\mathbf{T} \mid \mathbf{C} \mid \mathbf{G} \mid \mathbf{F}) \in [0, 1]^n, \quad \mathbf{T} \in [0, 1]^p, \mathbf{C} \in [0, 1]^c, \mathbf{G} \in [0, 1]^g, \mathbf{F} \in [0, 1]^f,$$

where $\mathbf{C}$ (contradiction) and $\mathbf{G}$ (ignorance) refine two kinds of indeterminacy.

Let t be a t -norm and s its dual t -conorm. For a valuation $\text{Val} : \text{Form}(\mathcal{L}) \rightarrow [0, 1]^n$, define, blockwise and componentwise,

$$\begin{aligned} \text{Val}(\phi \wedge \psi) &= (\mathbf{T}(\phi) \star_t \mathbf{T}(\psi) \mid \mathbf{C}(\phi) \star_s \mathbf{C}(\psi) \mid \mathbf{G}(\phi) \star_s \mathbf{G}(\psi) \mid \mathbf{F}(\phi) \star_s \mathbf{F}(\psi)), \\ \text{Val}(\phi \vee \psi) &= (\mathbf{T}(\phi) \star_s \mathbf{T}(\psi) \mid \mathbf{C}(\phi) \star_t \mathbf{C}(\psi) \mid \mathbf{G}(\phi) \star_t \mathbf{G}(\psi) \mid \mathbf{F}(\phi) \star_t \mathbf{F}(\psi)), \\ \text{Val}(\neg \phi) &= (\mathbf{F}(\phi) \mid \mathbf{C}(\phi) \mid \mathbf{G}(\phi) \mid \mathbf{T}(\phi)). \end{aligned}$$

Example 3.2 (Medical Diagnosis ($p=2, c=1, g=1, f=2; n=6$)). (cf.[60, 61]) **Blocks and semantics.**

$$\mathbf{v} = (T_1, T_2 \mid C \mid G \mid F_1, F_2) \in [0, 1]^6,$$

where T_1 (lab test) and T_2 (imaging) contribute to truth; C measures contradiction (e.g., conflicting clinicians); G captures ignorance/unknowns (e.g., missing data); F_1, F_2 quantify falsity evidence from the same sources (lab, imaging).

Atomic formulas. Let A be “Patient has pneumonia” and B be “Patient has bacterial infection.”

$$\text{Val}(A) = (0.82, 0.76 \mid 0.12 \mid 0.15 \mid 0.08, 0.10), \quad \text{Val}(B) = (0.70, 0.88 \mid 0.18 \mid 0.10 \mid 0.07, 0.12).$$

Conjunction and disjunction. With $t = \min$ and $s = \max$:

$$\begin{aligned} \text{Val}(A \wedge B) &= (\min(0.82, 0.70), \min(0.76, 0.88) \mid \max(0.12, 0.18) \\ &\quad \mid \max(0.15, 0.10) \mid \max(0.08, 0.07), \max(0.10, 0.12)) \\ &= (0.70, 0.76 \mid 0.18 \mid 0.15 \mid 0.08, 0.12), \\ \text{Val}(A \vee B) &= (\max(0.82, 0.70), \max(0.76, 0.88) \mid \min(0.12, 0.18) \\ &\quad \mid \min(0.15, 0.10) \mid \min(0.08, 0.07), \min(0.10, 0.12)) \\ &= (0.82, 0.88 \mid 0.12 \mid 0.10 \mid 0.07, 0.10). \end{aligned}$$

Negation. Swapping (T_1, T_2) with (F_1, F_2) and keeping (C, G) :

$$\text{Val}(\neg A) = (0.08, 0.10 \mid 0.12 \mid 0.15 \mid 0.82, 0.76).$$

The strong truth block and reduced falsity in $\text{Val}(A \wedge B)$ support a joint diagnosis; the disjunction preserves peak evidence while contracting C, G via $\min$.

Example 3.3 (Cyber Intrusion Detection ($p=2, c=1, g=1, f=2; n=6$)). (cf.[62, 63]) **Blocks and semantics.**

$$\mathbf{v} = (T_1, T_2 \mid C \mid G \mid F_1, F_2) \in [0, 1]^6,$$

where T_1 (network anomaly score) and T_2 (endpoint alert score) support truth; C is source-level contradiction; G denotes data gaps; F_1, F_2 are falsity evidence (e.g., benign baselines for network/endpoint).

Atomic formulas. Let C^* be “Ongoing intrusion present” and D^* be “Privileged-account compromise.”

$$\begin{aligned} \text{Val}(C^*) &= (0.80, 0.67 \mid 0.20 \mid 0.12 \mid 0.15, 0.22), \\ \text{Val}(D^*) &= (0.62, 0.74 \mid 0.25 \mid 0.18 \mid 0.20, 0.17). \end{aligned}$$

Conjunction and disjunction. With $t = \min$ and $s = \max$:

$$\begin{aligned} \text{Val}(C^* \wedge D^*) &= (\min(0.80, 0.62), \min(0.67, 0.74) \mid \max(0.20, 0.25) \mid \\ &\quad \max(0.12, 0.18) \mid \max(0.15, 0.20), \max(0.22, 0.17)) \\ &= (0.62, 0.67 \mid 0.25 \mid 0.18 \mid 0.20, 0.22), \\ \text{Val}(C^* \vee D^*) &= (\max(0.80, 0.62), \max(0.67, 0.74) \mid \min(0.20, 0.25) \\ &\quad \mid \min(0.12, 0.18) \mid \min(0.15, 0.20), \min(0.22, 0.17)) \\ &= (0.80, 0.74 \mid 0.20 \mid 0.12 \mid 0.15, 0.17). \end{aligned}$$

Negation.

$$\text{Val}(\neg C^*) = (0.15, 0.22 \mid 0.20 \mid 0.12 \mid 0.80, 0.67).$$

The $\wedge$ result shows conservative truth (via $\min$) with elevated contradiction/ignorance, appropriate for joint high-risk events; the $\vee$ result preserves the higher alerts and suppresses falsity.



Lemma 3.4 (Canonical merge and split between n-RQNL and n-RNL). *Let $r := c + g$. Define the merge (a surjection)*

$$\pi_Q : [0, 1]^{p+c+g+f} \longrightarrow [0, 1]^{p+r+f}, \quad \pi_Q(\mathbf{T} \mid \mathbf{C} \mid \mathbf{G} \mid \mathbf{F}) := (\mathbf{T} \mid \mathbf{C} \parallel \mathbf{G} \mid \mathbf{F}),$$

and for any fixed disjoint partition of indices $J_C \cup J_G = \{1, \dots, r\}$ with $|J_C| = c$, $|J_G| = g$, the split (an injection)

$$\sigma_Q : [0, 1]^{p+r+f} \longrightarrow [0, 1]^{p+c+g+f}, \quad \sigma_Q(\mathbf{T} \mid \mathbf{I} \mid \mathbf{F}) := (\mathbf{T} \mid \mathbf{I} \upharpoonright_{J_C} \mid \mathbf{I} \upharpoonright_{J_G} \mid \mathbf{F}).$$

Then $\pi_Q \circ \sigma_Q = \text{id}$.

Proof. Fix integers $p, c, g, f \geq 1$ and put $r := c + g$. By hypothesis we fix a disjoint partition $J_C \cup J_G = \{1, \dots, r\}$ with $|J_C| = c$, $|J_G| = g$, and we agree that for any $\mathbf{I} = (I_1, \dots, I_r) \in [0, 1]^r$ the restricted subvectors $\mathbf{I} \upharpoonright_{J_C} \in [0, 1]^c$ and $\mathbf{I} \upharpoonright_{J_G} \in [0, 1]^g$ are listed in the increasing order of indices in J_C and J_G , respectively. Concatenation $\parallel$ means simple blockwise juxtaposition: if $\mathbf{u} \in [0, 1]^c$ and $\mathbf{v} \in [0, 1]^g$ then $\mathbf{u} \parallel \mathbf{v} \in [0, 1]^r$ is the r -tuple obtained by putting $\mathbf{u}$ first and $\mathbf{v}$ second.

Claim. For every $(\mathbf{T} \mid \mathbf{I} \mid \mathbf{F}) \in [0, 1]^{p+r+f}$ one has

$$(\pi_Q \circ \sigma_Q)(\mathbf{T} \mid \mathbf{I} \mid \mathbf{F}) = (\mathbf{T} \mid \mathbf{I} \mid \mathbf{F}).$$

Coordinatewise verification. By definition of the split,

$$\sigma_Q(\mathbf{T} \mid \mathbf{I} \mid \mathbf{F}) = (\mathbf{T} \mid \mathbf{I} \upharpoonright_{J_C} \mid \mathbf{I} \upharpoonright_{J_G} \mid \mathbf{F}) \in [0, 1]^{p+c+g+f}.$$

Applying the merge π_Q gives

$$\pi_Q(\sigma_Q(\mathbf{T} \mid \mathbf{I} \mid \mathbf{F})) = (\mathbf{T} \mid \underbrace{\mathbf{I} \upharpoonright_{J_C} \parallel \mathbf{I} \upharpoonright_{J_G}}_{(*)} \mid \mathbf{F}) \in [0, 1]^{p+r+f}.$$

It remains to show that $(*) = \mathbf{I}$. Write $J_C = \{j_1^C < \dots < j_c^C\}$ and $J_G = \{j_1^G < \dots < j_g^G\}$. Then, by the ordering convention for restrictions,

$$\mathbf{I} \upharpoonright_{J_C} = (I_{j_1^C}, \dots, I_{j_c^C}), \quad \mathbf{I} \upharpoonright_{J_G} = (I_{j_1^G}, \dots, I_{j_g^G}),$$

and hence

$$\mathbf{I} \upharpoonright_{J_C} \parallel \mathbf{I} \upharpoonright_{J_G} = (I_{j_1^C}, \dots, I_{j_c^C}, I_{j_1^G}, \dots, I_{j_g^G}).$$

Because $\{j_1^C, \dots, j_c^C, j_1^G, \dots, j_g^G\} = \{1, \dots, r\}$ and the resulting r -tuple is *exactly* the chosen order of the r -block in $[0, 1]^{p+r+f}$ (first the J_C coordinates, then the J_G coordinates), we identify it with $\mathbf{I} = (I_1, \dots, I_r)$ under this fixed convention. Therefore $(*) = \mathbf{I}$, and consequently

$$(\pi_Q \circ \sigma_Q)(\mathbf{T} \mid \mathbf{I} \mid \mathbf{F}) = (\mathbf{T} \mid \mathbf{I} \mid \mathbf{F}).$$

Since this holds for all $(\mathbf{T} \mid \mathbf{I} \mid \mathbf{F})$, we obtain $\pi_Q \circ \sigma_Q = \text{id}_{[0,1]^{p+r+f}}$.

From $\pi_Q \circ \sigma_Q = \text{id}$ it follows immediately that σ_Q is injective (it has a left inverse) and π_Q is surjective (it has a right inverse), as stated. This completes the proof. $\square$

Theorem 3.5 (n-RQNL generalizes n-RNL and quadripartitioned logic). *With the connectives of Definitions 3.1 (n-RQNL) and of n-RNL above, for every t -norm t and its dual s :*

1. **(Reduction to n-RNL).** *The merge π_Q is an algebra homomorphism for $\wedge, \vee, \neg$, i.e.,*

$$\pi_Q(\text{Val}_Q(\circ)) = \text{Val}_N(\circ) \text{ on } [0, 1]^{p+r+f}, \quad \circ \in \{\wedge, \vee, \neg\},$$

where Val_Q (resp. Val_N) denotes an n-RQNL (resp. n-RNL) valuation and $r = c + g$. Consequently, n-RQNL reduces to n-RNL via π_Q .

2. **(Embedding of n-RNL).** *The split σ_Q is an algebra embedding: $\sigma_Q(\text{Val}_N(\circ)) = \text{Val}_Q(\circ)$ for $\circ \in \{\wedge, \vee, \neg\}$. Hence n-RQNL extends n-RNL.*
3. **(Unrefined case).** *For $p = c = g = f = 1$ the above block rules collapse to the usual quadripartitioned neutrosophic logic on scalars (T, C, G, F) ; therefore n-RQNL strictly generalizes the unrefined quadripartitioned logic.*

Proof. (1) Write $\text{Val}_Q(\phi) = (\mathbf{T}_\phi \mid \mathbf{C}_\phi \mid \mathbf{G}_\phi \mid \mathbf{F}_\phi)$. By definition of $\wedge$ in n-RQNL and n-RNL, and by componentwise lifting,

$$\pi_Q(\text{Val}_Q(\phi \wedge \psi)) = (\mathbf{T}_\phi \star_t \mathbf{T}_\psi \mid \underbrace{\mathbf{C}_\phi \star_s \mathbf{C}_\psi \parallel \mathbf{G}_\phi \star_s \mathbf{G}_\psi}_{\text{length } r} \mid \mathbf{F}_\phi \star_s \mathbf{F}_\psi).$$

In n-RNL, the indeterminacy block is length r and combines by s componentwise, so this equals $\text{Val}_N(\phi \wedge \psi)$ after π_Q 's concatenation. The $\vee$ case is analogous, using s on $\mathbf{T}$ and t on each indeterminacy component; $\neg$ follows from swapping $\mathbf{T} \leftrightarrow \mathbf{F}$ and keeping $\mathbf{C}, \mathbf{G}$ fixed, which under π_Q becomes the standard $(\mathbf{T} \mid \mathbf{I} \mid \mathbf{F}) \mapsto (\mathbf{F} \mid \mathbf{I} \mid \mathbf{T})$.

(2) For σ_Q , the split just relabels the r -dimensional indeterminacy vector into two subblocks without changing component values. Because the n-RNL and n-RQNL rules act componentwise by the *same* operator on each indeterminacy coordinate (s for $\wedge$, t for $\vee$), σ_Q commutes with all three connectives, hence is an embedding. Finally $\pi_Q \circ \sigma_Q = \text{id}$ by Lemma 3.4.

(3) Setting $p = c = g = f = 1$ turns each block into a scalar and reproduces the usual rules: T uses t/s for $\wedge/\vee$, while C, G, F use s/t for $\wedge/\vee$. The refined case strictly extends the unrefined because, e.g., with $c = 2$ one can distinguish two contradictory sources that the scalar C collapses. $\square$



3.2 Refined Pentapartitioned Neutrosophic Logic

Refined Pentapartitioned Neutrosophic Logic partitions evidence into truth, contradiction, ignorance, unknown, falsity, and combines blocks by designated t-norms and t-conorms.

Definition 3.6 (n-Refined Pentapartitioned Neutrosophic Logic (n-RPNL)). Fix integers $p, c, g, u, f \geq 1$ and put $n := p+c+g+u+f$. An n-refined pentapartitioned truth value is

$$\mathbf{v} = (\mathbf{T} \mid \mathbf{C} \mid \mathbf{G} \mid \mathbf{U} \mid \mathbf{F}) \in [0, 1]^n,$$

with block lengths p, c, g, u, f (truth, contradiction, ignorance, unknown, falsity). For a valuation Val , define (componentwise within each block), with the same t -norm t and t -conorm s :

$$\begin{aligned} \text{Val}(\phi \wedge \psi) &= (\mathbf{T}(\phi) \star_t \mathbf{T}(\psi) \mid \mathbf{C}(\phi) \star_s \mathbf{C}(\psi) \mid \mathbf{G}(\phi) \star_s \mathbf{G}(\psi) \mid \mathbf{U}(\phi) \star_s \mathbf{U}(\psi) \mid \mathbf{F}(\phi) \star_s \mathbf{F}(\psi)), \\ \text{Val}(\phi \vee \psi) &= (\mathbf{T}(\phi) \star_s \mathbf{T}(\psi) \mid \mathbf{C}(\phi) \star_t \mathbf{C}(\psi) \mid \mathbf{G}(\phi) \star_t \mathbf{G}(\psi) \mid \mathbf{U}(\phi) \star_t \mathbf{U}(\psi) \mid \mathbf{F}(\phi) \star_t \mathbf{F}(\psi)), \\ \text{Val}(\neg \phi) &= (\mathbf{F}(\phi) \mid \mathbf{C}(\phi) \mid \mathbf{G}(\phi) \mid \mathbf{U}(\phi) \mid \mathbf{T}(\phi)). \end{aligned}$$

For the block-swap negation to be an endomorphism of this fixed five-block space, we assume $p = f$. Otherwise, negation is interpreted as a typed map into the corresponding space with the truth and falsity block lengths exchanged.

Example 3.7 (Autonomous Driving: Pedestrian Stop Decision ($p=2, c=1, g=1, u=1, f=2; n=7$)). **Blocks and semantics.**

$$\mathbf{v} = (T_1, T_2 \mid C \mid G \mid U \mid F_1, F_2) \in [0, 1]^7,$$

where T_1 (camera) and T_2 (LiDAR) support truth; C quantifies contradiction (e.g. camera vs. LiDAR disagreement); G captures ignorance (missing/low-quality data); U represents unknown factors (unmodelled occlusions); and F_1, F_2 are falsity evidence (e.g. negative detectors for the two sensors).

Atomic propositions. Let A be “a pedestrian is present in the crosswalk” and B be “the crosswalk is currently occupied.” Assign:

$$\text{Val}(A) = (0.83, 0.78 \mid 0.12 \mid 0.10 \mid 0.08 \mid 0.07, 0.11), \quad \text{Val}(B) = (0.75, 0.81 \mid 0.15 \mid 0.09 \mid 0.06 \mid 0.10, 0.08).$$

Conjunction and disjunction. With $t = \min$ and $s = \max$:

$$\begin{aligned} \text{Val}(A \wedge B) &= (\min(0.83, 0.75), \min(0.78, 0.81) \mid \max(0.12, 0.15) \mid \max(0.10, 0.09) \mid \\ &\quad \max(0.08, 0.06) \mid \max(0.07, 0.10), \max(0.11, 0.08)) \\ &= (0.75, 0.78 \mid 0.15 \mid 0.10 \mid 0.08 \mid 0.10, 0.11), \\ \text{Val}(A \vee B) &= (\max(0.83, 0.75), \max(0.78, 0.81) \mid \min(0.12, 0.15) \mid \\ &\quad \min(0.10, 0.09) \mid \min(0.08, 0.06) \mid \min(0.07, 0.10), \min(0.11, 0.08)) \\ &= (0.83, 0.81 \mid 0.12 \mid 0.09 \mid 0.06 \mid 0.07, 0.08). \end{aligned}$$

Negation. Swapping truth and falsity blocks, keeping (C, G, U) :

$$\text{Val}(\neg A) = (0.07, 0.11 \mid 0.12 \mid 0.10 \mid 0.08 \mid 0.83, 0.78).$$

The conservative $\wedge$ pushes T down via $\min$ and raises (C, G, U) via $\max$, reflecting stricter joint evidence; the liberal $\vee$ preserves peak truth while shrinking indeterminacies through $\min$, indicating that evidence for either condition suffices to trigger caution.

Example 3.8 (Financial Security: Fraud and Account Takeover ($p=2, c=1, g=1, u=1, f=2; n=7$)). **Blocks and semantics.**

$$\mathbf{v} = (T_1, T_2 \mid C \mid G \mid U \mid F_1, F_2) \in [0, 1]^7,$$

where T_1 (supervised classifier) and T_2 (rule engine) support truth; C captures conflict between models; G encodes missing features; U collects unknown, unexplained anomalies; and F_1, F_2 quantify benign evidence (merchant whitelist, customer behavioral baseline).

Atomic propositions. Let C^* be “the transaction is fraudulent” and D^* be “the account is taken over.” Assign:

$$\begin{aligned} \text{Val}(C^*) &= (0.69, 0.77 \mid 0.22 \mid 0.18 \mid 0.12 \mid 0.25, 0.14), \\ \text{Val}(D^*) &= (0.74, 0.63 \mid 0.19 \mid 0.20 \mid 0.15 \mid 0.21, 0.17). \end{aligned}$$

Conjunction and disjunction. With $t = \min$ and $s = \max$:

$$\begin{aligned} \text{Val}(C^* \wedge D^*) &= (\min(0.69, 0.74), \min(0.77, 0.63) \mid \max(0.22, 0.19) \mid \max(0.18, 0.20) \mid \\ &\quad \max(0.12, 0.15) \mid \max(0.25, 0.21), \max(0.14, 0.17)) \\ &= (0.69, 0.63 \mid 0.22 \mid 0.20 \mid 0.15 \mid 0.25, 0.17), \\ \text{Val}(C^* \vee D^*) &= (\max(0.69, 0.74), \max(0.77, 0.63) \mid \min(0.22, 0.19) \mid \min(0.18, 0.20) \mid \\ &\quad \min(0.12, 0.15) \mid \min(0.25, 0.21), \min(0.14, 0.17)) \\ &= (0.74, 0.77 \mid 0.19 \mid 0.18 \mid 0.12 \mid 0.21, 0.14). \end{aligned}$$

Negation.

$$\text{Val}(\neg C^*) = (0.25, 0.14 \mid 0.22 \mid 0.18 \mid 0.12 \mid 0.69, 0.77).$$

The joint claim ($\wedge$) enforces conservative truth and elevates (C, G, U) , appropriate for simultaneous fraud and takeover. The alternative claim ($\vee$) keeps the strongest truth signals while damping falsity/indeterminacies, capturing “either risk suffices” alerting behavior.



Lemma 3.9 (Canonical merge and split between n-RPNL and n-RNL). *Let $r := c + g + u$. Define*

$$\pi_P(\mathbf{T} \mid \mathbf{C} \mid \mathbf{G} \mid \mathbf{U} \mid \mathbf{F}) := (\mathbf{T} \mid \mathbf{C} \parallel \mathbf{G} \parallel \mathbf{U} \mid \mathbf{F}),$$

and for any disjoint partition $J_C \cup J_G \cup J_U = \{1, \dots, r\}$ with $|J_C| = c$, $|J_G| = g$, $|J_U| = u$ define

$$\sigma_P(\mathbf{T} \mid \mathbf{I} \mid \mathbf{F}) := (\mathbf{T} \mid \mathbf{I} \upharpoonright_{J_C} \mid \mathbf{I} \upharpoonright_{J_G} \mid \mathbf{I} \upharpoonright_{J_U} \mid \mathbf{F}).$$

Then $\pi_P \circ \sigma_P = \text{id}$.

Proof. Fix $p, c, g, u, f \geq 1$ and put $r := c + g + u$. Let $(\mathbf{T} \mid \mathbf{I} \mid \mathbf{F}) \in [0, 1]^{p+r+f}$ be arbitrary, where $\mathbf{I} = (I_1, \dots, I_r)$. Apply the split:

$$\sigma_P(\mathbf{T} \mid \mathbf{I} \mid \mathbf{F}) = (\mathbf{T} \mid \mathbf{I} \upharpoonright_{J_C} \mid \mathbf{I} \upharpoonright_{J_G} \mid \mathbf{I} \upharpoonright_{J_U} \mid \mathbf{F}) \in [0, 1]^{p+c+g+u+f}.$$

Now apply the merge:

$$\pi_P(\sigma_P(\mathbf{T} \mid \mathbf{I} \mid \mathbf{F})) = (\mathbf{T} \mid \mathbf{I} \upharpoonright_{J_C} \parallel \mathbf{I} \upharpoonright_{J_G} \parallel \mathbf{I} \upharpoonright_{J_U} \mid \mathbf{F}).$$

Because $J_C \cup J_G \cup J_U = \{1, \dots, r\}$ is a disjoint partition and each restriction lists coordinates in increasing index order, the concatenation exactly reconstructs $\mathbf{I}$:

$$\mathbf{I} \upharpoonright_{J_C} \parallel \mathbf{I} \upharpoonright_{J_G} \parallel \mathbf{I} \upharpoonright_{J_U} = \mathbf{I}.$$

Hence $(\pi_P \circ \sigma_P)(\mathbf{T} \mid \mathbf{I} \mid \mathbf{F}) = (\mathbf{T} \mid \mathbf{I} \mid \mathbf{F})$ for all inputs, i.e. $\pi_P \circ \sigma_P = \text{id}_{[0,1]^{p+r+f}}$. In particular, σ_P is injective (it has a left inverse) and π_P is surjective (it has a right inverse). $\square$

Theorem 3.10 (n-RPNL generalizes n-RNL and pentapartitioned logic). *For the connectives in Definitions 3.6 and n-RNL, and any t, s :*

1. (**Reduction to n-RNL**). π_P is an algebra homomorphism for $\wedge, \vee, \neg$, hence n-RPNL reduces to n-RNL via π_P .
2. (**Embedding of n-RNL**). σ_P is an algebra embedding that commutes with $\wedge, \vee, \neg$, so n-RPNL extends n-RNL.
3. (**Unrefined case**). For $p = c = g = u = f = 1$ one recovers the classical pentapartitioned neutrosophic logic on scalars (T, C, G, U, F) ; thus the refined version strictly generalizes the unrefined one.

Proof. Identical to Theorem 3.5, using that in both n-RPNL and n-RNL the indeterminacy coordinates combine componentwise by s (for $\wedge$) and by t (for $\vee$). Concatenation (merge) and fixed index splits commute with componentwise application of the same binary operator by associativity and commutativity of t and s . Negation swaps only $\mathbf{T}$ and $\mathbf{F}$. $\square$

3.3 Refined Heptapartitioned Neutrosophic Logic

Refined Heptapartitioned Neutrosophic Logic represents truth, relative truth, contradiction, unknown information, ignorance, relative falsity, and falsity by separate vector-valued blocks. Its conjunction and disjunction operators are defined blockwise using a t -norm and its dual t -conorm, while negation exchanges the truth-like and falsity-like blocks and preserves the indeterminacy-related blocks.

Definition 3.11 (n-Refined Heptapartitioned Neutrosophic Logic (n-RHNL)). Fix integers $p, m, c, u, i, k, f \geq 1$ and set $n := p + m + c + u + i + k + f$. An n -refined heptapartitioned neutrosophic truth value is a 7-block vector

$$\mathbf{v} = (\mathbf{T} \mid \mathbf{M} \mid \mathbf{C} \mid \mathbf{U} \mid \mathbf{I} \mid \mathbf{K} \mid \mathbf{F}) \in [0, 1]^n,$$

with block lengths $\mathbf{T} \in [0, 1]^p$, $\mathbf{M} \in [0, 1]^m$, $\mathbf{C} \in [0, 1]^c$, $\mathbf{U} \in [0, 1]^u$, $\mathbf{I} \in [0, 1]^i$, $\mathbf{K} \in [0, 1]^k$, $\mathbf{F} \in [0, 1]^f$. Here $\mathbf{M}$ (relative truth) is truth-like, $\mathbf{K}$ (relative falsity) is falsity-like, and $\mathbf{C}, \mathbf{U}, \mathbf{I}$ refine three kinds of indeterminacy.

For the block-swap negation defined below to be an endomorphism of the same fixed seven-block space, we assume

$$p = f \quad \text{and} \quad m = k.$$

Thus, the truth/falsity blocks and the relative-truth/relative-falsity blocks occur in dual pairs of equal lengths. If these equalities are not imposed, the same formula for negation may instead be interpreted as a typed map into the corresponding space with the truth-like and falsity-like block lengths exchanged.

Let t be a (left-continuous) t -norm and s its dual t -conorm. A valuation $\text{Val} : \text{Form}(\mathcal{L}) \rightarrow [0, 1]^n$ assigns to each formula ϕ a 7-block vector, with the connectives defined *blockwise and componentwise* by

$$\begin{aligned} \text{Val}(\phi \wedge \psi) &= (\mathbf{T}(\phi) \star_t \mathbf{T}(\psi) \mid \mathbf{M}(\phi) \star_t \mathbf{M}(\psi) \mid \mathbf{C}(\phi) \star_s \mathbf{C}(\psi) \mid \\ &\quad \mathbf{U}(\phi) \star_s \mathbf{U}(\psi) \mid \mathbf{I}(\phi) \star_s \mathbf{I}(\psi) \mid \mathbf{K}(\phi) \star_s \mathbf{K}(\psi) \mid \mathbf{F}(\phi) \star_s \mathbf{F}(\psi)), \\ \text{Val}(\phi \vee \psi) &= (\mathbf{T}(\phi) \star_s \mathbf{T}(\psi) \mid \mathbf{M}(\phi) \star_s \mathbf{M}(\psi) \mid \mathbf{C}(\phi) \star_t \mathbf{C}(\psi) \mid \\ &\quad \mathbf{U}(\phi) \star_t \mathbf{U}(\psi) \mid \mathbf{I}(\phi) \star_t \mathbf{I}(\psi) \mid \mathbf{K}(\phi) \star_t \mathbf{K}(\psi) \mid \mathbf{F}(\phi) \star_t \mathbf{F}(\psi)), \\ \text{Val}(\neg \phi) &= (\mathbf{F}(\phi) \mid \mathbf{K}(\phi) \mid \mathbf{C}(\phi) \mid \\ &\quad \mathbf{U}(\phi) \mid \mathbf{I}(\phi) \mid \mathbf{M}(\phi) \mid \mathbf{T}(\phi)), \end{aligned}$$

i.e., negation swaps truth-like with falsity-like blocks ($\mathbf{T} \leftrightarrow \mathbf{F}$, $\mathbf{M} \leftrightarrow \mathbf{K}$) and leaves the three indeterminacy blocks invariant. (If a strong negation is desired, one may also apply $x \mapsto 1 - x$ componentwise within the swapped blocks.)



Example 3.12 (n–RHNL in Autonomous–Driving Perception (p=2, m=1, c=u=i=1, k=1, f=2; n = 9)). **Blocks and semantics.**

$$\mathbf{v} = (\underbrace{T_1, T_2}_{\text{truth}} \mid \underbrace{M}_{\text{relative truth}} \mid \underbrace{C}_{\text{contradiction}} \mid \underbrace{U}_{\text{unknown}} \mid \underbrace{I}_{\text{ignorance}} \mid \underbrace{K}_{\text{relative falsity}} \mid \underbrace{F_1, F_2}_{\text{falsity}})$$

Interpretation: T_1 (camera), T_2 (LiDAR) truth; M (contextual/relative truth); C contradiction; U unknown; I ignorance; K relative falsity; F_1, F_2 falsity (camera/LiDAR).

Atomic formulas. Let A be “Pedestrian is in the crosswalk” and B be “Traffic light is red”. Take the valuations (all in $[0, 1]$):

$$\text{Val}(A) = (0.85, 0.70 \mid 0.60 \mid 0.10 \mid 0.20 \mid 0.15 \mid 0.25 \mid 0.05, 0.10),$$

$$\text{Val}(B) = (0.90, 0.65 \mid 0.55 \mid 0.05 \mid 0.10 \mid 0.10 \mid 0.30 \mid 0.05, 0.15).$$

Connectives. With $t = \min$, $s = \max$, apply Definition 3.11 blockwise:

$$\begin{aligned} \text{Val}(A \wedge B) &= (\min(0.85, 0.90), \min(0.70, 0.65) \mid \min(0.60, 0.55) \mid \max(0.10, 0.05) \mid \\ &\quad \max(0.20, 0.10) \mid \max(0.15, 0.10) \mid \max(0.25, 0.30) \mid \max(0.05, 0.05), \max(0.10, 0.15)) \\ &= (0.85, 0.65 \mid 0.55 \mid 0.10 \mid 0.20 \mid 0.15 \mid 0.30 \mid 0.05, 0.15), \end{aligned}$$

$$\begin{aligned} \text{Val}(A \vee B) &= (\max(0.85, 0.90), \max(0.70, 0.65) \mid \max(0.60, 0.55) \mid \min(0.10, 0.05) \mid \\ &\quad \min(0.20, 0.10) \mid \min(0.15, 0.10) \mid \min(0.25, 0.30) \mid \min(0.05, 0.05), \min(0.10, 0.15)) \\ &= (0.90, 0.70 \mid 0.60 \mid 0.05 \mid 0.10 \mid 0.10 \mid 0.25 \mid 0.05, 0.10). \end{aligned}$$

Negation and implication. Negation swaps $(T_1, T_2) \leftrightarrow (F_1, F_2)$ and $M \leftrightarrow K$:

$$\text{Val}(\neg A) = (0.05, 0.10 \mid 0.25 \mid 0.10 \mid 0.20 \mid 0.15 \mid 0.60 \mid 0.85, 0.70).$$

Define implication by $A \Rightarrow B := \neg A \vee B$. Then

$$\begin{aligned} \text{Val}(A \Rightarrow B) &= \text{Val}(\neg A \vee B) \\ &= (\max(0.05, 0.90), \max(0.10, 0.65) \mid \max(0.25, 0.55) \mid \min(0.10, 0.05) \mid \min(0.20, 0.10) \\ &\quad \mid \min(0.15, 0.10) \mid \min(0.60, 0.30) \mid \min(0.85, 0.05), \min(0.70, 0.15)) \\ &= (0.90, 0.65 \mid 0.55 \mid 0.05 \mid 0.10 \mid 0.10 \mid 0.30 \mid 0.05, 0.15). \end{aligned}$$

Reading. The high truth block and low falsity in $\text{Val}(A \Rightarrow B)$ indicate a strong support for “if pedestrian detected then light is red,” moderated by small C, U, I .

Example 3.13 (n–RHNL in Historical Source Authentication (p=1, m=2, c=1, u=1, i=2, k=2, f=1; n = 10)). **Blocks and semantics.**

$$\mathbf{v} = (\underbrace{T}_{\text{objective truth}} \mid \underbrace{M_1, M_2}_{\text{relative/contextual truth}} \mid \underbrace{C}_{\text{contradiction}} \mid \underbrace{U}_{\text{unknown}} \mid \underbrace{I_1, I_2}_{\text{ignorance types}} \mid \underbrace{K_1, K_2}_{\text{relative falsity}} \mid \underbrace{F}_{\text{objective falsity}})$$

Interpretation: T (e.g., carbon dating), (M_1, M_2) (paleography, provenance), C (conflicting testimony), U (gaps), (I_1, I_2) (measurement limits, ambiguous style), (K_1, K_2) (contextual counter–evidence), F (hard falsification).

Atomic formulas. Let H be “Document D is authentic” and G be “Document D is forged”.

$$\text{Val}(H) = (0.72 \mid 0.65, 0.55 \mid 0.20 \mid 0.10 \mid 0.25, 0.30 \mid 0.35, 0.40 \mid 0.18),$$

$$\text{Val}(G) = (0.30 \mid 0.40, 0.45 \mid 0.25 \mid 0.15 \mid 0.20, 0.25 \mid 0.50, 0.55 \mid 0.60).$$

Negation. With $p = f = 1$ and $m = k = 2$, negation is blockwise well–typed:

$$\text{Val}(\neg G) = (0.60 \mid 0.50, 0.55 \mid 0.25 \mid 0.15 \mid 0.20, 0.25 \mid 0.40, 0.45 \mid 0.30).$$

Conjunction $H \wedge \neg G$. Using $t = \min$, $s = \max$,

$$\begin{aligned} \text{Val}(H \wedge \neg G) &= (\min(0.72, 0.60) \mid \min(0.65, 0.50), \min(0.55, 0.55) \mid \max(0.20, 0.25) \mid \\ &\quad \max(0.10, 0.15) \mid \max(0.25, 0.20), \max(0.30, 0.25) \mid \max(0.35, 0.40), \\ &\quad \max(0.40, 0.45) \mid \max(0.18, 0.30)) \\ &= (0.60 \mid 0.50, 0.55 \mid 0.25 \mid 0.15 \mid 0.25, 0.30 \mid 0.40, 0.45 \mid 0.30). \end{aligned}$$

Disjunction $H \vee G$.

$$\begin{aligned} \text{Val}(H \vee G) &= (\max(0.72, 0.30) \mid \max(0.65, 0.40), \max(0.55, 0.45) \mid \min(0.20, 0.25) \mid \\ &\quad \min(0.10, 0.15) \mid \min(0.25, 0.20), \min(0.30, 0.25) \mid \min(0.35, 0.50), \\ &\quad \min(0.40, 0.55) \mid \min(0.18, 0.60)) \\ &= (0.72 \mid 0.65, 0.55 \mid 0.20 \mid 0.10 \mid 0.20, 0.25 \mid 0.35, 0.40 \mid 0.18). \end{aligned}$$

The strong T and M in $\text{Val}(H \wedge \neg G)$, with controlled C, U, I and moderate (K_1, K_2, F) , support authenticity given current evidence. The disjunction $\text{Val}(H \vee G)$ preserves the higher truth–like components while contracting falsity–like ones via t –norms.

Remark 3.14. The optional “sum–boundedness” constraint $\sum(\text{all components}) \in [0, n]$ can be imposed independently; it is preserved by the above operators since $t, s : [0, 1]^2 \rightarrow [0, 1]$ and the rules act componentwise.



Lemma 3.15 (Merge/Split between n-RHNL and n-RQNL). Let $r_Q := c + (u + i)$ and $p_Q := p + m$, $f_Q := f + k$. Define the merge

$$\pi_Q : [0, 1]^{p+m+c+u+i+k+f} \rightarrow [0, 1]^{p_Q+r_Q+f_Q}, \quad \pi_Q(\mathbf{T} \mid \mathbf{M} \mid \mathbf{C} \mid \mathbf{U} \mid \mathbf{I} \mid \mathbf{K} \mid \mathbf{F}) := ((\mathbf{T} \parallel \mathbf{M}) \mid \mathbf{C} \parallel (\mathbf{U} \parallel \mathbf{I}) \mid (\mathbf{F} \parallel \mathbf{K})).$$

Fix any disjoint index split of the r_Q -dimensional middle block into J_C of size c and J_G of size $u + i$; also split the truth/falsity aggregates into J_T of size p and J_M of size m , and J_F of size f and J_K of size k . Define the split

$$\sigma_Q(\mathbf{T} \mid \mathbf{G} \mid \mathbf{F}) := (\mathbf{T} \upharpoonright_{J_T} \mid \mathbf{T} \upharpoonright_{J_M} \mid \mathbf{G} \upharpoonright_{J_C} \mid \mathbf{G} \upharpoonright_{J_G} \mid \mathbf{0}_i \mid \mathbf{F} \upharpoonright_{J_K} \mid \mathbf{F} \upharpoonright_{J_F}),$$

where $\mathbf{0}_i$ is the all-zeros vector used only if one chooses to keep U and I distinguished (alternatively, distribute J_G further into U and I with sizes u and i respectively). Then $\pi_Q \circ \sigma_Q = \text{id}$.

Proof. Put $r_Q := c + (u + i)$, $p_Q := p + m$, and $f_Q := f + k$. Fix disjoint index partitions

$$J_T \cup J_M = \{1, \dots, p_Q\}, \quad J_C \cup J_G = \{1, \dots, r_Q\}, \quad J_F \cup J_K = \{1, \dots, f_Q\},$$

with $|J_T| = p$, $|J_M| = m$, $|J_C| = c$, $|J_G| = u + i$, $|J_F| = f$, $|J_K| = k$. (For the middle block, further split J_G as $J_G = J_U \cup J_I$ with $|J_U| = u$ and $|J_I| = i$; this is the “distributed” option mentioned in the lemma.)

Let $(\mathbf{T} \mid \mathbf{G} \mid \mathbf{F}) \in [0, 1]^{p_Q+r_Q+f_Q}$ be arbitrary. By definition of the split we take

$$\sigma_Q(\mathbf{T} \mid \mathbf{G} \mid \mathbf{F}) = (\mathbf{T} \upharpoonright_{J_T} \mid \mathbf{T} \upharpoonright_{J_M} \mid \mathbf{G} \upharpoonright_{J_C} \mid \mathbf{G} \upharpoonright_{J_U} \mid \mathbf{G} \upharpoonright_{J_I} \mid \mathbf{F} \upharpoonright_{J_K} \mid \mathbf{F} \upharpoonright_{J_F}) \in [0, 1]^{p+m+c+u+i+k+f}.$$

Now apply the merge π_Q :

$$\pi_Q(\sigma_Q(\mathbf{T} \mid \mathbf{G} \mid \mathbf{F})) = ((\mathbf{T} \upharpoonright_{J_T}) \parallel (\mathbf{T} \upharpoonright_{J_M}) \mid (\mathbf{G} \upharpoonright_{J_C}) \parallel ((\mathbf{G} \upharpoonright_{J_U}) \parallel (\mathbf{G} \upharpoonright_{J_I})) \mid (\mathbf{F} \upharpoonright_{J_F}) \parallel (\mathbf{F} \upharpoonright_{J_K})).$$

Since $J_T \cup J_M$, $J_C \cup J_U \cup J_I$, and $J_F \cup J_K$ are disjoint unions covering their respective index sets (and each restriction lists coordinates in increasing index order), we have the equalities of r -tuples

$$(\mathbf{T} \upharpoonright_{J_T}) \parallel (\mathbf{T} \upharpoonright_{J_M}) = \mathbf{T}, \quad (\mathbf{G} \upharpoonright_{J_C}) \parallel (\mathbf{G} \upharpoonright_{J_U}) \parallel (\mathbf{G} \upharpoonright_{J_I}) = \mathbf{G}, \quad (\mathbf{F} \upharpoonright_{J_F}) \parallel (\mathbf{F} \upharpoonright_{J_K}) = \mathbf{F}.$$

Therefore

$$(\pi_Q \circ \sigma_Q)(\mathbf{T} \mid \mathbf{G} \mid \mathbf{F}) = (\mathbf{T} \mid \mathbf{G} \mid \mathbf{F}),$$

for all $(\mathbf{T} \mid \mathbf{G} \mid \mathbf{F})$, i.e. $\pi_Q \circ \sigma_Q = \text{id}_{[0,1]^{p_Q+r_Q+f_Q}}$. Consequently, σ_Q is injective (it has a left inverse) and π_Q is surjective (it has a right inverse). $\square$

Theorem 3.16 (n-RHNL generalizes n-RQNL). Let Val_H be an n-RHNL valuation and Val_Q an n-RQNL valuation defined with the same t and s . Then the merge π_Q is a homomorphism for $\wedge, \vee, \neg$:

$$\pi_Q(\text{Val}_H(\circ)) = \text{Val}_Q(\circ) \quad \text{for } \circ \in \{\wedge, \vee, \neg\}.$$

Consequently, n-RHNL reduces to n-RQNL under π_Q and strictly extends it via the split σ_Q .

Proof. Write $\text{Val}_H(\phi) = (\mathbf{T}_\phi \mid \mathbf{M}_\phi \mid \mathbf{C}_\phi \mid \mathbf{U}_\phi \mid \mathbf{I}_\phi \mid \mathbf{K}_\phi \mid \mathbf{F}_\phi)$. For $\wedge$, the $\mathbf{T}$ and $\mathbf{M}$ blocks combine by t (truth-like), while all $\mathbf{C}, \mathbf{U}, \mathbf{I}, \mathbf{K}, \mathbf{F}$ combine by s . After concatenation by π_Q , the first block $(\mathbf{T} \parallel \mathbf{M})$ is t -combined componentwise, matching the T -block of n-RQNL. The middle block $\mathbf{C} \parallel (\mathbf{U} \parallel \mathbf{I})$ is s -combined, matching the two indeterminacy subblocks in n-RQNL (both use s for $\wedge$). The last block $(\mathbf{F} \parallel \mathbf{K})$ is s -combined, matching the F -block in n-RQNL. The $\vee$ case is analogous with $t \leftrightarrow s$ on the corresponding blocks. For $\neg$, swapping $(\mathbf{T}, \mathbf{M}) \leftrightarrow (\mathbf{F}, \mathbf{K})$ and fixing $(\mathbf{C}, \mathbf{U}, \mathbf{I})$ maps under π_Q to the standard $(T \mid G \mid F) \mapsto (F \mid G \mid T)$. Thus π_Q is a homomorphism. Since σ_Q refines blocks without changing component values (or splits them by fixed indices), it commutes with componentwise operators and gives an embedding. Strictness follows because different distributions between $\mathbf{M}$ and $\mathbf{T}$ (or between $\mathbf{K}$ and $\mathbf{F}$, or among $\mathbf{U}, \mathbf{I}$) collapse under π_Q . $\square$

Lemma 3.17 (Merge/Split between n-RHNL and n-RPNL). Let $p_P := p + m$, $f_P := f + k$, and $r_P := c + u + i$. Define

$$\pi_P(\mathbf{T} \mid \mathbf{M} \mid \mathbf{C} \mid \mathbf{U} \mid \mathbf{I} \mid \mathbf{K} \mid \mathbf{F}) := ((\mathbf{T} \parallel \mathbf{M}) \mid \mathbf{C} \mid \mathbf{I} \mid \mathbf{U} \mid (\mathbf{F} \parallel \mathbf{K})) \in [0, 1]^{p_P+c+i+u+f_P}.$$

Conversely, fix index splits of the first and last blocks into (J_T, J_M) and (J_F, J_K) with sizes (p, m) and (f, k) , and split the indeterminacy middle into J_C, J_I, J_U with sizes (c, i, u) . Then define σ_P by placing the corresponding coordinates back into $(\mathbf{T}, \mathbf{M}, \mathbf{C}, \mathbf{U}, \mathbf{I}, \mathbf{K}, \mathbf{F})$ in that order. One has $\pi_P \circ \sigma_P = \text{id}$.

Proof. Put $p_P := p + m$, $f_P := f + k$, $r_P := c + u + i$. Fix disjoint index partitions

$$J_T \cup J_M = \{1, \dots, p_P\}, \quad J_F \cup J_K = \{1, \dots, f_P\},$$

with $|J_T| = p$, $|J_M| = m$, $|J_F| = f$, $|J_K| = k$. (Each restriction lists coordinates in increasing index order.)

Let $(\mathbf{T} \mid \mathbf{C} \mid \mathbf{I} \mid \mathbf{U} \mid \mathbf{F}) \in [0, 1]^{p_P+c+i+u+f_P}$ be arbitrary. By definition of the split map σ_P (placing coordinates back into the heptapartitioned order $(\mathbf{T}, \mathbf{M}, \mathbf{C}, \mathbf{U}, \mathbf{I}, \mathbf{K}, \mathbf{F})$), we set

$$\sigma_P(\mathbf{T} \mid \mathbf{C} \mid \mathbf{I} \mid \mathbf{U} \mid \mathbf{F}) = (\mathbf{T} \upharpoonright_{J_T} \mid \mathbf{T} \upharpoonright_{J_M} \mid \mathbf{C} \mid \mathbf{U} \mid \mathbf{I} \mid \mathbf{F} \upharpoonright_{J_K} \mid \mathbf{F} \upharpoonright_{J_F}) \in [0, 1]^{p+m+c+u+i+k+f}.$$

Now apply the merge π_P :

$$\pi_P(\sigma_P(\mathbf{T} \mid \mathbf{C} \mid \mathbf{I} \mid \mathbf{U} \mid \mathbf{F})) = ((\mathbf{T} \upharpoonright_{J_T}) \parallel (\mathbf{T} \upharpoonright_{J_M}) \mid \mathbf{C} \mid \mathbf{I} \mid \mathbf{U} \mid (\mathbf{F} \upharpoonright_{J_F}) \parallel (\mathbf{F} \upharpoonright_{J_K})).$$

Because $J_T \cup J_M$ and $J_F \cup J_K$ are disjoint partitions of their index sets, we have

$$(\mathbf{T} \upharpoonright_{J_T}) \parallel (\mathbf{T} \upharpoonright_{J_M}) = \mathbf{T}, \quad (\mathbf{F} \upharpoonright_{J_F}) \parallel (\mathbf{F} \upharpoonright_{J_K}) = \mathbf{F}.$$

Hence

$$(\pi_P \circ \sigma_P)(\mathbf{T} \mid \mathbf{C} \mid \mathbf{I} \mid \mathbf{U} \mid \mathbf{F}) = (\mathbf{T} \mid \mathbf{C} \mid \mathbf{I} \mid \mathbf{U} \mid \mathbf{F}),$$

for all inputs, i.e. $\pi_P \circ \sigma_P = \text{id}_{[0,1]^{p_P+c+i+u+f_P}}$. In particular, σ_P is injective (it has a left inverse) and π_P is surjective (it has a right inverse). $\square$



Theorem 3.18 (Compatibility of n-RHNL and n-RPNL under reblocking). Assume $p = f$ and $m = k$, and let the n-RHNL and n-RPNL valuations be defined using the same t -norm t and its dual t -conorm s . Then the merge map

$$\pi_P$$

is a homomorphism for

$$\wedge, \vee, \neg.$$

Hence the n-RHNL operations are compatible with the corresponding n-RPNL operations under the reblocking map π_P .

Moreover, for a fixed split whose truth/falsity index partitions are compatible with the duality induced by negation, the associated split map σ_P is an injective homomorphism.

Proof. Let

$$\mathbf{v} = (\mathbf{T}_v \mid \mathbf{M}_v \mid \mathbf{C}_v \mid \mathbf{U}_v \mid \mathbf{I}_v \mid \mathbf{K}_v \mid \mathbf{F}_v)$$

and

$$\mathbf{w} = (\mathbf{T}_w \mid \mathbf{M}_w \mid \mathbf{C}_w \mid \mathbf{U}_w \mid \mathbf{I}_w \mid \mathbf{K}_w \mid \mathbf{F}_w)$$

be two n-RHNL truth values.

For clarity, write

$$\wedge_H, \vee_H, \neg_H$$

for the n-RHNL operations and

$$\wedge_P, \vee_P, \neg_P$$

for the corresponding n-RPNL operations.

Recall that

$$\pi_P(\mathbf{v}) = (\mathbf{T}_v \parallel \mathbf{M}_v \mid \mathbf{C}_v \mid \mathbf{I}_v \mid \mathbf{U}_v \mid \mathbf{F}_v \parallel \mathbf{K}_v),$$

where the block $\mathbf{I}_v$ is interpreted as the ignorance block $\mathbf{G}$ of n-RPNL.

For any componentwise binary operator $\star_q$, where $q \in \{t, s\}$, concatenation satisfies

$$(\mathbf{a} \star_q \mathbf{b}) \parallel (\mathbf{c} \star_q \mathbf{d}) = (\mathbf{a} \parallel \mathbf{c}) \star_q (\mathbf{b} \parallel \mathbf{d}),$$

whenever the corresponding block lengths agree.

For conjunction, the truth-like blocks $\mathbf{T}, \mathbf{M}$ are combined componentwise by t , whereas $\mathbf{C}, \mathbf{U}, \mathbf{I}, \mathbf{K}, \mathbf{F}$ are combined componentwise by s . Therefore,

$$\begin{aligned} \pi_P(\mathbf{v} \wedge_H \mathbf{w}) &= \left((\mathbf{T}_v \star_t \mathbf{T}_w) \parallel (\mathbf{M}_v \star_t \mathbf{M}_w) \mid \mathbf{C}_v \star_s \mathbf{C}_w \right. \\ &\quad \left. \mid \mathbf{I}_v \star_s \mathbf{I}_w \mid \mathbf{U}_v \star_s \mathbf{U}_w \right. \\ &\quad \left. \mid (\mathbf{F}_v \star_s \mathbf{F}_w) \parallel (\mathbf{K}_v \star_s \mathbf{K}_w) \right) \\ &= \left((\mathbf{T}_v \parallel \mathbf{M}_v) \star_t (\mathbf{T}_w \parallel \mathbf{M}_w) \mid \mathbf{C}_v \star_s \mathbf{C}_w \right. \\ &\quad \left. \mid \mathbf{I}_v \star_s \mathbf{I}_w \mid \mathbf{U}_v \star_s \mathbf{U}_w \right. \\ &\quad \left. \mid (\mathbf{F}_v \parallel \mathbf{K}_v) \star_s (\mathbf{F}_w \parallel \mathbf{K}_w) \right) \\ &= \pi_P(\mathbf{v}) \wedge_P \pi_P(\mathbf{w}). \end{aligned}$$

For disjunction, the truth-like blocks are combined by s , whereas the remaining blocks are combined by t . Hence,

$$\begin{aligned} \pi_P(\mathbf{v} \vee_H \mathbf{w}) &= \left((\mathbf{T}_v \parallel \mathbf{M}_v) \star_s (\mathbf{T}_w \parallel \mathbf{M}_w) \mid \mathbf{C}_v \star_t \mathbf{C}_w \right. \\ &\quad \left. \mid \mathbf{I}_v \star_t \mathbf{I}_w \mid \mathbf{U}_v \star_t \mathbf{U}_w \right. \\ &\quad \left. \mid (\mathbf{F}_v \parallel \mathbf{K}_v) \star_t (\mathbf{F}_w \parallel \mathbf{K}_w) \right) \\ &= \pi_P(\mathbf{v}) \vee_P \pi_P(\mathbf{w}). \end{aligned}$$

Finally, using the n-RHNL negation,

$$\neg_H \mathbf{v} = (\mathbf{F}_v \mid \mathbf{K}_v \mid \mathbf{C}_v \mid \mathbf{U}_v \mid \mathbf{I}_v \mid \mathbf{M}_v \mid \mathbf{T}_v),$$

we obtain

$$\begin{aligned} \pi_P(\neg_H \mathbf{v}) &= (\mathbf{F}_v \parallel \mathbf{K}_v \mid \mathbf{C}_v \mid \mathbf{I}_v \mid \mathbf{U}_v \mid \mathbf{T}_v \parallel \mathbf{M}_v) \\ &= \neg_P(\mathbf{T}_v \parallel \mathbf{M}_v \mid \mathbf{C}_v \mid \mathbf{I}_v \mid \mathbf{U}_v \mid \mathbf{F}_v \parallel \mathbf{K}_v) \\ &= \neg_P \pi_P(\mathbf{v}). \end{aligned}$$

Thus,

$$\begin{aligned} \pi_P(\mathbf{v} \wedge_H \mathbf{w}) &= \pi_P(\mathbf{v}) \wedge_P \pi_P(\mathbf{w}), \\ \pi_P(\mathbf{v} \vee_H \mathbf{w}) &= \pi_P(\mathbf{v}) \vee_P \pi_P(\mathbf{w}), \end{aligned}$$

and

$$\pi_P(\neg_H \mathbf{v}) = \neg_P \pi_P(\mathbf{v}).$$



Hence π_P is a homomorphism for $\wedge, \vee, \neg$.

The relation

$$\pi_P \circ \sigma_P = \text{id}$$

proved in Lemma 3.17 implies that σ_P is injective. Furthermore, restrictions to fixed index blocks commute with the componentwise t -norm and t -conorm operations. Thus σ_P preserves $\wedge$ and $\vee$. If the truth/falsity index partitions are chosen compatibly with the duality used by negation, then σ_P also preserves $\neg$. Therefore, under this compatibility assumption, σ_P is an algebra embedding. $\square$

Theorem 3.19 (Recovery of (scalar) Heptapartitioned Neutrosophic Logic). *Setting all block lengths to 1 ($p = m = c = u = i = k = f = 1$) turns each block into a scalar $(T, M, C, U, I, K, F) \in [0, 1]^7$. Then the operators of Definition 3.11 reduce to*

$$\begin{aligned} \wedge : (T, M; C, U, I; K, F) &\mapsto (t(T, T'), t(M, M'); s(C, C'), s(U, U'), s(I, I'); s(K, K'), s(F, F')), \\ \vee : (T, M; C, U, I; K, F) &\mapsto (s(T, T'), s(M, M'); t(C, C'), t(U, U'), t(I, I'); t(K, K'), t(F, F')), \\ \neg : (T, M; C, U, I; K, F) &\mapsto (F, K; C, U, I; M, T), \end{aligned}$$

which is the standard (unrefined) heptapartitioned behavior: truth-like (T, M) act dually to falsity-like (F, K) , while the three indeterminacy facets (C, U, I) are fixed by negation and behave conjunctively/disjunctively via s/t . Therefore n -RHNL specializes to Heptapartitioned Neutrosophic Logic.

Proof. Immediate by substituting block length 1 and reading Definition 3.11 componentwise. $\square$

3.4 Iterative Refined Neutrosophic Logic (Refined of ... of Refined)

Iterative Refined Neutrosophic Logic (IRNL) provides a refinement schema that repeatedly decomposes truth-like, indeterminacy-like, and falsity-like components. Its logical connectives are defined using a t -norm and its dual t -conorm, and its semantics remain stable under successive refinement steps.

Definition 3.20 (t -norm / t -conorm). A t -norm is a map $t : [0, 1]^2 \rightarrow [0, 1]$ that is associative, commutative, monotone in each argument, and satisfies $t(x, 1) = x$. Its (De Morgan) dual t -conorm $s : [0, 1]^2 \rightarrow [0, 1]$ is associative, commutative, monotone, with $s(x, 0) = x$. Typical choices are $t = \min$ with $s = \max$, or product $t(x, y) = xy$ with $s(x, y) = x + y - xy$.

Definition 3.21 (Refinement schema). An iterative refinement schema is a tuple

$$\Sigma = (\mathcal{T}, \mathcal{I}, \mathcal{F}, \delta),$$

where $\mathcal{T}, \mathcal{I}, \mathcal{F}$ are finite, pairwise disjoint index sets of truth-like, indeterminacy-like, and falsity-like leaves, and $\delta : \mathcal{T} \xrightarrow{\cong} \mathcal{F}$ is a fixed bijection (the duality pairing). Write $L := \mathcal{T} \sqcup \mathcal{I} \sqcup \mathcal{F}$ and

$$\mathcal{V}_\Sigma := [0, 1]^L = \left\{ v = (v_\ell)_{\ell \in L} : v_\ell \in [0, 1] \right\}.$$

For $v \in \mathcal{V}_\Sigma$ we denote its restrictions by $v|_{\mathcal{T}}, v|_{\mathcal{I}}, v|_{\mathcal{F}}$.

Example 3.22 (Refinement schema in a smart-thermostat decision). **Scenario.** A smart thermostat decides whether to heat now. We build a refinement schema

$$\Sigma = (\mathcal{T}, \mathcal{I}, \mathcal{F}, \delta),$$

where the leaves and the duality pairing are

$$\begin{aligned} \mathcal{T} &= \{\text{T}_{\text{dmd}}, \text{T}_{\text{wthr}}\}, \quad \mathcal{I} = \{\text{I}_{\text{sens}}\}, \quad \mathcal{F} = \{\text{F}_{\text{sun}}, \text{F}_{\text{occ}}\}, \\ \delta(\text{T}_{\text{dmd}}) &= \text{F}_{\text{sun}}, \quad \delta(\text{T}_{\text{wthr}}) = \text{F}_{\text{occ}}. \end{aligned}$$

Here, T_{dmd} = “heat-demand from setpoint gap,” T_{wthr} = “cold weather,” I_{sens} = “sensor uncertainty,” F_{sun} = “incoming solar gain,” and F_{occ} = “low occupancy (discourages heating).”

A concrete state. On a winter morning, the thermostat aggregates signals into

$$v \in \mathcal{V}_\Sigma = [0, 1]^{\mathcal{T} \sqcup \mathcal{I} \sqcup \mathcal{F}}, \quad v = (0.78, 0.65 \mid 0.12 \mid 0.30, 0.10),$$

ordered as $(\text{T}_{\text{dmd}}, \text{T}_{\text{wthr}} \mid \text{I}_{\text{sens}} \mid \text{F}_{\text{sun}}, \text{F}_{\text{occ}})$. Thus $v|_{\mathcal{T}} = (0.78, 0.65)$, $v|_{\mathcal{I}} = (0.12)$, $v|_{\mathcal{F}} = (0.30, 0.10)$. The duality δ pairs truth-like demand with solar counter-evidence and truth-like weather with occupancy counter-evidence, preparing for negation in IRNL.

Definition 3.23 (Iterative Refined Neutrosophic Logic (IRNL)). Fix a language of formulas Form , a refinement schema Σ as in Definition 3.21, and a pair (t, s) consisting of a t -norm and its dual t -conorm. An IRNL valuation is a map

$$\text{Val} : \text{Form} \longrightarrow \mathcal{V}_\Sigma,$$

together with connectives defined coordinatewise on L by:

$$\begin{aligned} \text{Conjunction:} \quad (\text{Val}(\phi \wedge \psi))_\ell &= \begin{cases} t(\text{Val}(\phi)_\ell, \text{Val}(\psi)_\ell), & \ell \in \mathcal{T} \quad (\text{truth-like}); \\ s(\text{Val}(\phi)_\ell, \text{Val}(\psi)_\ell), & \ell \in \mathcal{I} \cup \mathcal{F} \quad (\text{indet./falsity-like}). \end{cases} \\ \text{Disjunction:} \quad (\text{Val}(\phi \vee \psi))_\ell &= \begin{cases} s(\text{Val}(\phi)_\ell, \text{Val}(\psi)_\ell), & \ell \in \mathcal{T}; \\ t(\text{Val}(\phi)_\ell, \text{Val}(\psi)_\ell), & \ell \in \mathcal{I} \cup \mathcal{F}. \end{cases} \\ \text{Negation:} \quad (\text{Val}(\neg \phi))_\ell &= \begin{cases} \text{Val}(\phi)_{\delta^{-1}(\ell)}, & \ell \in \mathcal{F} \quad (\text{swap truth} \leftrightarrow \text{falsity}); \\ \text{Val}(\phi)_{\delta(\ell)}, & \ell \in \mathcal{T}; \\ \text{Val}(\phi)_\ell, & \ell \in \mathcal{I} \quad (\text{indeterminacy fixed}). \end{cases} \end{aligned}$$

(If a strong negation is desired, additionally apply $x \mapsto 1 - x$ in the swapped coordinates.)



Example 3.24 (IRNL on the same schema with explicit calculations). **Fix** the Gödel pair $(t, s) = (\min, \max)$. Consider two formulas:

$$\phi := \text{“Heat now is appropriate”}, \quad \psi := \text{“A cold front is imminent”}.$$

Using the schema of Example 3.22, assign valuations

$$\text{Val}(\phi) = (0.78, 0.65 \mid 0.12 \mid 0.30, 0.10), \quad \text{Val}(\psi) = (0.55, 0.80 \mid 0.15 \mid 0.20, 0.05).$$

Conjunction $\phi \wedge \psi$ (truth-like by min, indeterminacy/falsity-like by max):

$$\mathbf{T} = (\min(0.78, 0.55), \min(0.65, 0.80)) = (0.55, 0.65),$$

$$\mathbf{I} = \max(0.12, 0.15) = 0.15, \quad \mathbf{F} = (\max(0.30, 0.20), \max(0.10, 0.05)) = (0.30, 0.10),$$

hence

$$\text{Val}(\phi \wedge \psi) = (0.55, 0.65 \mid 0.15 \mid 0.30, 0.10).$$

Disjunction $\phi \vee \psi$ (truth-like by max, indeterminacy/falsity-like by min):

$$\mathbf{T} = (\max(0.78, 0.55), \max(0.65, 0.80)) = (0.78, 0.80),$$

$$\mathbf{I} = \min(0.12, 0.15) = 0.12, \quad \mathbf{F} = (\min(0.30, 0.20), \min(0.10, 0.05)) = (0.20, 0.05),$$

so

$$\text{Val}(\phi \vee \psi) = (0.78, 0.80 \mid 0.12 \mid 0.20, 0.05).$$

Negation $\neg\phi$ (swap via δ ; $\mathbf{T}_{\text{dmd}} \leftrightarrow \mathbf{F}_{\text{sun}}, \mathbf{T}_{\text{wthr}} \leftrightarrow \mathbf{F}_{\text{occ}}$; keep $\mathbf{I}_{\text{sens}}$):

$$\text{Val}(\neg\phi) = \left(\underbrace{0.30, 0.10}_{\text{from } (\mathbf{F}_{\text{sun}}, \mathbf{F}_{\text{occ}})} \mid 0.12 \mid \underbrace{0.78, 0.65}_{\text{from } (\mathbf{T}_{\text{dmd}}, \mathbf{T}_{\text{wthr}})} \right).$$

These computations make explicit the coordinatewise action of IRNL on a real household control task.

Definition 3.25 (One refinement step). Let $\Sigma = (\mathcal{T}, \mathcal{I}, \mathcal{F}, \delta)$ be a schema. Choose a finite family of disjoint leaf blocks $B \subseteq \mathcal{T} \sqcup \mathcal{I} \sqcup \mathcal{F}$ to be refined. For each $\beta \in B$ select an integer $k(\beta) \geq 1$ and create *children indices* $\beta^{(1)}, \dots, \beta^{(k(\beta))}$ of the *same* type as β . Obtain a new schema

$$\Sigma' = (\mathcal{T}', \mathcal{I}', \mathcal{F}', \delta')$$

by replacing each β with its children inside the appropriate set ($\mathcal{T}$ or $\mathcal{I}$ or $\mathcal{F}$), and extend the duality pairing by refining *paired leaves symmetrically*: if $\beta \in \mathcal{T}$ is refined, also refine $\delta(\beta) \in \mathcal{F}$ with the same $k(\beta)$, and set $\delta'(\beta^{(j)}) := \delta(\beta)^{(j)}$ for $j = 1, \dots, k(\beta)$. (Leaves not in B and their pairings are kept unchanged.)

Example 3.26 (One refinement step: splitting a weather leaf and its dual). Starting from Σ in Example 3.22, choose the refinement set

$$B = \{\mathbf{T}_{\text{wthr}}, \mathbf{F}_{\text{occ}}\},$$

and set $k(\beta) = 2$ for each $\beta \in B$. Create children

$$\mathbf{T}_{\text{wthr}}^{(1)} = \mathbf{T}_{\text{wthrA}} \text{ (forecast model A)}, \quad \mathbf{T}_{\text{wthr}}^{(2)} = \mathbf{T}_{\text{wthrB}} \text{ (forecast model B)},$$

$$\mathbf{F}_{\text{occ}}^{(1)} = \mathbf{F}_{\text{cal}} \text{ (calendar non-occupancy)}, \quad \mathbf{F}_{\text{occ}}^{(2)} = \mathbf{F}_{\text{mot}} \text{ (no motion detected)}.$$

Define the refined schema $\Sigma' = (\mathcal{T}', \mathcal{I}', \mathcal{F}', \delta')$ by replacing $\mathbf{T}_{\text{wthr}}$ with its two children in $\mathcal{T}$, and $\mathbf{F}_{\text{occ}}$ with its two children in $\mathcal{F}$, keeping other leaves unchanged. Extend the duality pairing *symmetrically*:

$$\delta'(\mathbf{T}_{\text{wthrA}}) = \mathbf{F}_{\text{cal}}, \quad \delta'(\mathbf{T}_{\text{wthrB}}) = \mathbf{F}_{\text{mot}},$$

and keep $\delta'(\mathbf{T}_{\text{dmd}}) = \mathbf{F}_{\text{sun}}$, δ' unchanged on non-refined pairs. This respects Definition 3.25 by refining a truth-leaf and its falsity-dual with the same arity.

Definition 3.27 (Split map (canonical embedding)). The refinement step $\Sigma \rightsquigarrow \Sigma'$ induces a *split* (or *refinement*) map $\sigma : \mathcal{V}_{\Sigma} \rightarrow \mathcal{V}_{\Sigma'}$ defined by

$$(\sigma(v))_{\beta^{(j)}} := v_{\beta} \quad \text{for every refined } \beta \in B \text{ and } j = 1, \dots, k(\beta), \quad (\sigma(v))_{\ell} := v_{\ell} \text{ for all non-refined } \ell.$$

Thus each refined coordinate is *replicated* across its children.

Example 3.28 (Split map (canonical embedding) with explicit numbers). Let $v = (0.78, 0.65 \mid 0.12 \mid 0.30, 0.10) \in \mathcal{V}_{\Sigma}$ be as in Example 3.22, with ordering $(\mathbf{T}_{\text{dmd}}, \mathbf{T}_{\text{wthr}} \mid \mathbf{I}_{\text{sens}} \mid \mathbf{F}_{\text{sun}}, \mathbf{F}_{\text{occ}})$. The split map $\sigma : \mathcal{V}_{\Sigma} \rightarrow \mathcal{V}_{\Sigma'}$ from Example 3.26 replicates each refined coordinate across its children:

$$(\sigma(v))_{\mathbf{T}_{\text{wthrA}}} = (\sigma(v))_{\mathbf{T}_{\text{wthrB}}} = v_{\mathbf{T}_{\text{wthr}}} = 0.65, \quad (\sigma(v))_{\mathbf{F}_{\text{cal}}} = (\sigma(v))_{\mathbf{F}_{\text{mot}}} = v_{\mathbf{F}_{\text{occ}}} = 0.10,$$

and leaves other coordinates unchanged:

$$(\sigma(v))_{\mathbf{T}_{\text{dmd}}} = 0.78, \quad (\sigma(v))_{\mathbf{I}_{\text{sens}}} = 0.12, \quad (\sigma(v))_{\mathbf{F}_{\text{sun}}} = 0.30.$$

Thus, in the ordered block

$$(\mathbf{T}_{\text{dmd}}, \mathbf{T}_{\text{wthrA}}, \mathbf{T}_{\text{wthrB}} \mid \mathbf{I}_{\text{sens}} \mid \mathbf{F}_{\text{sun}}, \mathbf{F}_{\text{cal}}, \mathbf{F}_{\text{mot}}),$$

the embedded vector is

$$\sigma(v) = (0.78, 0.65, 0.65 \mid 0.12 \mid 0.30, 0.10, 0.10),$$

which satisfies Definition 3.27 exactly: every refined child inherits its parent's value, and all non-refined leaves are preserved verbatim.



Theorem 3.29 (Functoriality under refinement). *Let $\text{Val} : \text{Form} \rightarrow \mathcal{V}_\Sigma$ be an IRNL valuation and $\sigma : \mathcal{V}_\Sigma \rightarrow \mathcal{V}_{\Sigma'}$ the split map of Definition 3.27. Define $\text{Val}' := \sigma \circ \text{Val} : \text{Form} \rightarrow \mathcal{V}_{\Sigma'}$. Then for all formulas ϕ, ψ ,*

$$\text{Val}'(\phi \wedge \psi) = \text{Val}'(\phi) \wedge \text{Val}'(\psi), \quad \text{Val}'(\phi \vee \psi) = \text{Val}'(\phi) \vee \text{Val}'(\psi), \quad \text{Val}'(\neg\phi) = \neg \text{Val}'(\phi),$$

where the connectives on $\mathcal{V}_{\Sigma'}$ are those of Definition 3.23 built from (t, s) and the refined schema Σ' . In particular, refining after evaluating equals evaluating after refining.

Proof. All operations are componentwise. If β is refined, both $\text{Val}'(\phi)$ and $\text{Val}'(\psi)$ assign the same parent values v_β, w_β duplicated across $\beta^{(1)}, \dots, \beta^{(k)}$. For $\wedge$, each child coordinate computes either $t(v_\beta, w_\beta)$ (truth-like) or $s(v_\beta, w_\beta)$ (indet./falsity-like), which is exactly the duplication of the parent result; hence $\sigma(\text{Val}(\phi \wedge \psi)) = \text{Val}'(\phi) \wedge \text{Val}'(\psi)$. The $\vee$ case is identical with $t \leftrightarrow s$ on the corresponding types. For $\neg$, the symmetric refinement of paired leaves guarantees that swapping via δ' acts child-by-child as the swap via δ followed by replication. Thus $\sigma(\text{Val}(\neg\phi)) = \neg \text{Val}'(\phi)$. Non-refined coordinates are untouched in all cases. $\square$

Corollary 3.30 (Iterative (“refined of refined”) consistency). *By repeating Definition 3.25 a finite number of times, one obtains a chain*

$$\Sigma_0 \rightsquigarrow \Sigma_1 \rightsquigarrow \dots \rightsquigarrow \Sigma_m,$$

with split maps $\sigma_j : \mathcal{V}_{\Sigma_{j-1}} \rightarrow \mathcal{V}_{\Sigma_j}$. The composite $\sigma_1 \circ \dots \circ \sigma_m$ is a homomorphism for $\wedge, \vee, \neg$; hence the IRNL semantics is stable under arbitrary finite iterations of refinement.

Remark 3.31 (Relation to n -refined logics). If $\mathcal{T}, \mathcal{I}, \mathcal{F}$ are partitioned into blocks of sizes p, r, s respectively (single refinement level), then $\mathcal{V}_\Sigma \cong [0, 1]^{p+r+s}$ and Definition 3.23 specializes to the usual n -refined neutrosophic semantics (with $n = p+r+s$). The iterative scheme above permits arbitrarily many subsequent splits of any coordinate while preserving the connective behavior.

In addition to the examples discussed so far, two further concrete examples are provided below.

Example 3.32 (IRNL in Medical Diagnosis with One Refinement Step). **Schema.** Let the refinement schema be

$$\mathcal{T} = \{\text{T}_{\text{lab}}, \text{T}_{\text{img}}\}, \quad \mathcal{I} = \{\text{I}_{\text{amb}}, \text{I}_{\text{miss}}\}, \quad \mathcal{F} = \{\text{F}_{\text{lab}}, \text{F}_{\text{img}}\},$$

with the duality pairing $\delta(\text{T}_{\text{lab}}) = \text{F}_{\text{lab}}$ and $\delta(\text{T}_{\text{img}}) = \text{F}_{\text{img}}$. We fix $t = \min$ and $s = \max$ and order coordinates as

$$(\text{T}_{\text{lab}}, \text{T}_{\text{img}} \mid \text{I}_{\text{amb}}, \text{I}_{\text{miss}} \mid \text{F}_{\text{lab}}, \text{F}_{\text{img}}).$$

Atoms. Let p mean “Disease A is present” and q mean “Disease B is present”. Suppose the current evidential degrees are:

$$\text{Val}(p) = (0.70, 0.60 \mid 0.20, 0.10 \mid 0.10, 0.20), \quad \text{Val}(q) = (0.50, 0.80 \mid 0.10, 0.30 \mid 0.20, 0.10).$$

Connectives (before refinement).

$$\begin{aligned} \text{Val}(p \wedge q) &= (\min(0.70, 0.50), \min(0.60, 0.80) \mid \max(0.20, 0.10), \max(0.10, 0.30) \mid \\ &\quad \max(0.10, 0.20), \max(0.20, 0.10)) \\ &= (0.50, 0.60 \mid 0.20, 0.30 \mid 0.20, 0.20), \\ \text{Val}(p \vee q) &= (\max(0.70, 0.50), \max(0.60, 0.80) \mid \min(0.20, 0.10), \min(0.10, 0.30) \mid \\ &\quad \min(0.10, 0.20), \min(0.20, 0.10)) \\ &= (0.70, 0.80 \mid 0.10, 0.10 \mid 0.10, 0.10), \\ \text{Val}(\neg p) &= (\text{F}_{\text{lab}}, \text{F}_{\text{img}} \mid \text{I}_{\text{amb}}, \text{I}_{\text{miss}} \mid \\ &\quad \text{T}_{\text{lab}}, \text{T}_{\text{img}})\text{-swap of } \text{Val}(p) \\ &= (0.10, 0.20 \mid 0.20, 0.10 \mid 0.70, 0.60). \end{aligned}$$

Iterative refinement step. Refine the truth–falsity pair $(\text{T}_{\text{lab}}, \text{F}_{\text{lab}})$ into two children each:

$$\text{T}_{\text{lab}} \rightsquigarrow \text{T}_{\text{lab}}^{(1)}, \text{T}_{\text{lab}}^{(2)}, \quad \text{F}_{\text{lab}} \rightsquigarrow \text{F}_{\text{lab}}^{(1)}, \text{F}_{\text{lab}}^{(2)},$$

and extend the pairing by $\delta(\text{T}_{\text{lab}}^{(j)}) = \text{F}_{\text{lab}}^{(j)}$ ($j = 1, 2$). The split map σ replicates the parent coordinate into its children:

$$\sigma(\text{Val}(p)) = (0.70, 0.70, 0.60 \mid 0.20, 0.10 \mid 0.10, 0.10, 0.20),$$

in the refined order $(\text{T}_{\text{lab}}^{(1)}, \text{T}_{\text{lab}}^{(2)}, \text{T}_{\text{img}} \mid \text{I}_{\text{amb}}, \text{I}_{\text{miss}} \mid \text{F}_{\text{lab}}^{(1)}, \text{F}_{\text{lab}}^{(2)}, \text{F}_{\text{img}})$, and similarly

$$\sigma(\text{Val}(q)) = (0.50, 0.50, 0.80 \mid 0.10, 0.30 \mid 0.20, 0.20, 0.10).$$

Connectives (after refinement). Using the same $t = \min, s = \max$ componentwise,

$$\begin{aligned} \text{Val}'(p \wedge q) &= \sigma(\text{Val}(p)) \wedge \sigma(\text{Val}(q)) \\ &= (\min(0.70, 0.50), \min(0.70, 0.50), \min(0.60, 0.80) \mid \max(0.20, 0.10), \max(0.10, 0.30) \mid \\ &\quad \max(0.10, 0.20), \max(0.10, 0.20), \max(0.20, 0.10)) \\ &= (0.50, 0.50, 0.60 \mid 0.20, 0.30 \mid 0.20, 0.20, 0.20). \end{aligned}$$

On the other hand,

$$\sigma(\text{Val}(p \wedge q)) = \sigma(0.50, 0.60 \mid 0.20, 0.30 \mid 0.20, 0.20) = (0.50, 0.50, 0.60 \mid 0.20, 0.30 \mid 0.20, 0.20, 0.20),$$

which coincides with $\text{Val}'(p \wedge q)$, illustrating the refinement invariance $\sigma(\text{Val}(\cdot)) = \text{Val}'(\cdot)$ for $\wedge$. The same equality holds for $\vee$ and $\neg$ (omitted for brevity).



Example 3.33 (IRNL in Autonomous Driving with Indeterminacy Refinement). **Schema.** Let

$$\mathcal{T} = \{T_{\text{cam}}, T_{\text{lid}}, T_{\text{rad}}\}, \quad \mathcal{I} = \{I_{\text{occl}}, I_{\text{weather}}\}, \quad \mathcal{F} = \{F_{\text{cam}}, F_{\text{lid}}, F_{\text{rad}}\},$$

paired as $\delta(T_x) = F_x$ for $x \in \{\text{cam}, \text{lid}, \text{rad}\}$. Again we take $t = \min$, $s = \max$ and order coordinates as

$$(T_{\text{cam}}, T_{\text{lid}}, T_{\text{rad}} \mid I_{\text{occl}}, I_{\text{weather}} \mid F_{\text{cam}}, F_{\text{lid}}, F_{\text{rad}}).$$

Atoms. Let o denote “Obstacle ahead” and l denote “Left lane is free”. Assume

$$\text{Val}(o) = (0.80, 0.90, 0.60 \mid 0.30, 0.20 \mid 0.10, 0.05, 0.20),$$

$$\text{Val}(l) = (0.40, 0.30, 0.50 \mid 0.20, 0.40 \mid 0.40, 0.50, 0.30).$$

Negation swaps truth/falsity blocks (same pairing) and keeps indeterminacy:

$$\text{Val}(\neg l) = (0.40, 0.50, 0.30 \mid 0.20, 0.40 \mid 0.40, 0.30, 0.50).$$

A compound: $o \wedge \neg l$ (“obstacle ahead and not left-free”):

$$\begin{aligned} \text{Val}(o \wedge \neg l) &= \left(\min(0.80, 0.40), \min(0.90, 0.50), \min(0.60, 0.30) \mid \max(0.30, 0.20), \max(0.20, 0.40) \mid \right. \\ &\quad \left. \max(0.10, 0.40), \max(0.05, 0.30), \max(0.20, 0.50) \right) \\ &= (0.40, 0.50, 0.30 \mid 0.30, 0.40 \mid 0.40, 0.30, 0.50). \end{aligned}$$

Iterative refinement step (on indeterminacy). Split I_{occl} into two children of the same type:

$$I_{\text{occl}} \rightsquigarrow I_{\text{occl}}^{(s)} \text{ (static occlusion), } \quad I_{\text{occl}}^{(d)} \text{ (dynamic occlusion),}$$

keeping all other leaves unchanged (no dual pairing is needed for $\mathcal{I}$). The split map σ duplicates the occlusion value:

$$\sigma(\text{Val}(o)) = (0.80, 0.90, 0.60 \mid 0.30, 0.30, 0.20 \mid 0.10, 0.05, 0.20),$$

$$\sigma(\text{Val}(\neg l)) = (0.40, 0.50, 0.30 \mid 0.20, 0.20, 0.40 \mid 0.40, 0.30, 0.50),$$

where the refined order is

$$(T_{\text{cam}}, T_{\text{lid}}, T_{\text{rad}} \mid I_{\text{occl}}^{(s)}, I_{\text{occl}}^{(d)}, I_{\text{weather}} \mid F_{\text{cam}}, F_{\text{lid}}, F_{\text{rad}}).$$

Connective after refinement.

$$\begin{aligned} \text{Val}'(o \wedge \neg l) &= \sigma(\text{Val}(o)) \wedge \sigma(\text{Val}(\neg l)) \\ &= \left(\min(0.80, 0.40), \min(0.90, 0.50), \min(0.60, 0.30) \mid \right. \\ &\quad \left. \max(0.30, 0.20), \max(0.30, 0.20), \max(0.20, 0.40) \mid \right. \\ &\quad \left. \max(0.10, 0.40), \max(0.05, 0.30), \max(0.20, 0.50) \right) \\ &= (0.40, 0.50, 0.30 \mid 0.30, 0.30, 0.40 \mid 0.40, 0.30, 0.50). \end{aligned}$$

Applying σ to the pre-refinement result gives

$$\sigma(\text{Val}(o \wedge \neg l)) = (0.40, 0.50, 0.30 \mid 0.30, 0.30, 0.40 \mid 0.40, 0.30, 0.50),$$

which matches $\text{Val}'(o \wedge \neg l)$ exactly. Hence the evaluation commutes with the refinement step also when only indeterminacy leaves are split.

4 Additional Results: Iterated Refined Uncertain Sets

4.1 Iterated Refined Uncertain Sets

Uncertain Sets provide a unified framework capable of generalizing a wide variety of uncertainty-based concepts, including Fuzzy Sets, Neutrosophic Sets, and related structures [64, 65, 24, 66]. By introducing the notions of Refined Uncertain Sets and Iterated Refined Uncertain Sets, it is possible to extend this unifying perspective further and to encompass refined and hierarchically structured versions of existing uncertainty models. In this section, we investigate these concepts in detail. We first recall the general notion of an Uncertain Set and then introduce its refined and iteratively refined extensions.

Definition 4.1 (Uncertain Set). [64, 65] Let X be a nonempty universe, and let M be a fixed uncertainty model whose degree domain is

$$\text{Dom}(M) \subseteq [0, 1]^k$$

for some integer $k \geq 1$.

An *Uncertain Set of type M on X* , abbreviated as an M -Uncertain Set, is a mapping

$$\mu_A : X \longrightarrow \text{Dom}(M).$$



Equivalently, it may be written as

$$A = \{ \langle x, \mu_A(x) \rangle \mid x \in X \}.$$

For each $x \in X$, the value

$$\mu_A(x) = (\mu_A^1(x), \mu_A^2(x), \dots, \mu_A^k(x)) \in \text{Dom}(M)$$

is called the M -uncertainty degree or M -membership degree of x in A .

Different uncertainty models are obtained by choosing different degree domains. For example:

$$\text{Dom}(M) = [0, 1]$$

gives a fuzzy set,

$$\text{Dom}(M) \subseteq [0, 1]^2$$

may represent an intuitionistic fuzzy set, and

$$\text{Dom}(M) \subseteq [0, 1]^3$$

may represent a neutrosophic set.

Definition 4.2 (Refinement Profile and Refined Degree Domain). Let

$$\text{Dom}(M) \subseteq [0, 1]^k$$

be the degree domain of an uncertainty model M .

A *refinement profile* is a vector

$$\mathbf{r} = (r_1, r_2, \dots, r_k) \in \mathbb{N}_{\geq 1}^k,$$

where r_i specifies the number of refined subcomponents associated with the i -th original uncertainty component.

For each $i \in \{1, \dots, k\}$, let

$$\alpha_i : [0, 1]^{r_i} \longrightarrow [0, 1]$$

be an aggregation map. The corresponding aggregate refinement map is defined by

$$\Pi_{\mathbf{r}} : \prod_{i=1}^k [0, 1]^{r_i} \longrightarrow [0, 1]^k$$

as

$$\Pi_{\mathbf{r}}(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k) = (\alpha_1(\mathbf{u}_1), \alpha_2(\mathbf{u}_2), \dots, \alpha_k(\mathbf{u}_k)),$$

where

$$\mathbf{u}_i = (u_{i,1}, u_{i,2}, \dots, u_{i,r_i}) \in [0, 1]^{r_i}.$$

The *refined degree domain* associated with $(M, \mathbf{r})$ is

$$\text{Dom}_{\mathbf{r}}(M) = \left\{ (\mathbf{u}_1, \dots, \mathbf{u}_k) \in \prod_{i=1}^k [0, 1]^{r_i} \mid \Pi_{\mathbf{r}}(\mathbf{u}_1, \dots, \mathbf{u}_k) \in \text{Dom}(M) \right\}.$$

Definition 4.3 (Refined Uncertain Set). Let X be a nonempty universe, let M be an uncertainty model with

$$\text{Dom}(M) \subseteq [0, 1]^k,$$

and let

$$\mathbf{r} = (r_1, \dots, r_k) \in \mathbb{N}_{\geq 1}^k$$

be a refinement profile.

A *Refined Uncertain Set of type* $(M, \mathbf{r})$ on X is a mapping

$$\tilde{\mu}_A : X \longrightarrow \text{Dom}_{\mathbf{r}}(M).$$

For each $x \in X$, the refined uncertainty degree is written as

$$\tilde{\mu}_A(x) = \left(\mu_{A,1}^{(1)}(x), \dots, \mu_{A,r_1}^{(1)}(x) \mid \mu_{A,1}^{(2)}(x), \dots, \mu_{A,r_2}^{(2)}(x) \mid \dots \mid \mu_{A,1}^{(k)}(x), \dots, \mu_{A,r_k}^{(k)}(x) \right).$$

The block

$$\left(\mu_{A,1}^{(i)}(x), \dots, \mu_{A,r_i}^{(i)}(x) \right)$$

refines the i -th uncertainty component into r_i distinguishable subcomponents. These subcomponents may represent different information sources, criteria, experts, sensors, semantic categories, priority levels, or contextual conditions.

The associated coarse or unrefined membership degree is obtained by aggregation:

$$\mu_A(x) = \Pi_{\mathbf{r}}(\tilde{\mu}_A(x)).$$

Explicitly,

$$\mu_A(x) = \left(\alpha_1 \left(\mu_{A,1}^{(1)}(x), \dots, \mu_{A,r_1}^{(1)}(x) \right), \dots, \alpha_k \left(\mu_{A,1}^{(k)}(x), \dots, \mu_{A,r_k}^{(k)}(x) \right) \right).$$

If A is a prescribed M -Uncertain Set with membership function μ_A , then $\tilde{A}$ is called a *refinement of* A when

$$\mu_A = \Pi_{\mathbf{r}} \circ \tilde{\mu}_A.$$

Thus, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\tilde{\mu}_A} & \text{Dom}_{\mathbf{r}}(M) \\ \text{id}_X \downarrow & & \downarrow \Pi_{\mathbf{r}} \\ X & \xrightarrow{\mu_A} & \text{Dom}(M). \end{array}$$



Remark 4.4. If

$$r_1 = r_2 = \dots = r_k = 1$$

and every aggregation map satisfies

$$\alpha_i(u) = u,$$

then

$$\text{Dom}_{\mathbf{r}}(M) = \text{Dom}(M),$$

and a Refined Uncertain Set reduces to an ordinary Uncertain Set. Consequently, Uncertain Sets are special cases of Refined Uncertain Sets.

Remark 4.5 (Refined Neutrosophic Case). For a neutrosophic uncertainty model,

$$\text{Dom}(M) \subseteq [0, 1]^3,$$

the three original components correspond to truth, indeterminacy, and falsity. A refinement profile

$$\mathbf{r} = (p, q, r)$$

produces a refined degree of the form

$$\tilde{\mu}_A(x) = \left(T_1(x), \dots, T_p(x) \mid I_1(x), \dots, I_q(x) \mid F_1(x), \dots, F_r(x) \right).$$

Hence, Refined Neutrosophic Sets arise as a special case of Refined Uncertain Sets.

Definition 4.6 (Iterated Refined Uncertain Set). Let X be a nonempty universe, and let M be an uncertainty model with initial degree domain

$$D^{(0)} := \text{Dom}(M).$$

Fix a depth

$$L \in \mathbb{N}_{\geq 1}.$$

For each level

$$\ell \in \{0, 1, \dots, L-1\},$$

let $D^{(\ell+1)}$ be a refined degree domain of $D^{(\ell)}$, and let

$$\pi_\ell : D^{(\ell+1)} \longrightarrow D^{(\ell)}$$

be the corresponding aggregation or coarsening map.

An *Iterated Refined Uncertain Set of depth L* on X is a family of mappings

$$\mu_A^{(\ell)} : X \longrightarrow D^{(\ell)}, \quad \ell = 0, 1, \dots, L,$$

satisfying the compatibility conditions

$$\mu_A^{(\ell)} = \pi_\ell \circ \mu_A^{(\ell+1)}$$

for every

$$\ell \in \{0, 1, \dots, L-1\}.$$

Equivalently, for every $x \in X$,

$$\mu_A^{(\ell)}(x) = \pi_\ell \left(\mu_A^{(\ell+1)}(x) \right).$$

Thus, each degree at level ℓ is obtained by aggregating the more detailed degree at level $\ell + 1$. In particular,

$$\mu_A^{(0)} = \pi_0 \circ \pi_1 \circ \dots \circ \pi_{L-1} \circ \mu_A^{(L)}.$$

The complete iterated structure may be written as

$$\mathcal{I}A = \left(X, \{D^{(\ell)}\}_{\ell=0}^L, \{\pi_\ell\}_{\ell=0}^{L-1}, \{\mu_A^{(\ell)}\}_{\ell=0}^L \right).$$

Here,

$$\mu_A^{(0)} : X \longrightarrow \text{Dom}(M)$$

is the ordinary Uncertain Set,

$$\mu_A^{(1)}$$

is its first refinement,

$$\mu_A^{(2)}$$

is a refinement of the first refinement, and so forth.

Remark 4.7 (Commutative Refinement Chain). An Iterated Refined Uncertain Set determines the commutative chain

$$X \xrightarrow{\mu_A^{(L)}} D^{(L)} \xrightarrow{\pi_{L-1}} D^{(L-1)} \xrightarrow{\pi_{L-2}} \dots \xrightarrow{\pi_1} D^{(1)} \xrightarrow{\pi_0} D^{(0)}.$$

For every pair of levels $0 \leq i < j \leq L$, define

$$\pi_{i:j} = \pi_i \circ \pi_{i+1} \circ \dots \circ \pi_{j-1}.$$

Then

$$\mu_A^{(i)} = \pi_{i:j} \circ \mu_A^{(j)}.$$

Hence, every finer level consistently determines all coarser levels.



Definition 4.8 (Infinitely Iterated Refined Uncertain Set). Let

$$D^{(0)} \longleftarrow D^{(1)} \longleftarrow D^{(2)} \longleftarrow \dots$$

be an inverse system of uncertainty degree domains with aggregation maps

$$\pi_\ell : D^{(\ell+1)} \longrightarrow D^{(\ell)}.$$

An *Infinitely Iterated Refined Uncertain Set* on X is a sequence

$$\left\{ \mu_A^{(\ell)} : X \longrightarrow D^{(\ell)} \right\}_{\ell \geq 0}$$

such that

$$\mu_A^{(\ell)} = \pi_\ell \circ \mu_A^{(\ell+1)}$$

for every $\ell \geq 0$.

Equivalently, for each $x \in X$, the sequence

$$\left(\mu_A^{(0)}(x), \mu_A^{(1)}(x), \mu_A^{(2)}(x), \dots \right)$$

is an element of the inverse limit

$$\varprojlim_\ell D^{(\ell)} = \left\{ (d_0, d_1, d_2, \dots) \in \prod_{\ell \geq 0} D^{(\ell)} \mid \pi_\ell(d_{\ell+1}) = d_\ell \text{ for every } \ell \geq 0 \right\}.$$

Therefore, an Infinitely Iterated Refined Uncertain Set may equivalently be regarded as a mapping

$$\mu_A^{(\infty)} : X \longrightarrow \varprojlim_\ell D^{(\ell)}.$$

Remark 4.9. The distinction among the three notions can be summarized as follows:

$$\begin{aligned} \text{Uncertain Set} & : X \longrightarrow D^{(0)}, \\ \text{Refined Uncertain Set} & : X \longrightarrow D^{(1)} \longrightarrow D^{(0)}, \\ \text{Iterated Refined Uncertain Set} & : X \longrightarrow D^{(L)} \longrightarrow \dots \longrightarrow D^{(1)} \longrightarrow D^{(0)}. \end{aligned}$$

Thus, an ordinary Uncertain Set assigns one model-dependent uncertainty degree to each element; a Refined Uncertain Set decomposes that degree into interpretable subcomponents; and an Iterated Refined Uncertain Set repeats this decomposition through multiple compatible levels.

A comparison of Uncertain Sets, Refined Uncertain Sets, and Iterated Refined Uncertain Sets is presented in Table 6.

Table 6: Comparison of Uncertain, Refined Uncertain, and Iterated Refined Uncertain Sets

Aspect	Uncertain Set	Refined Uncertain Set	Iterated Refined Uncertain Set
Basic form	$\mu_A : X \rightarrow D^{(0)}$	$\tilde{\mu}_A : X \rightarrow D^{(1)}$	$\mu_A^{(\ell)} : X \rightarrow D^{(\ell)}, 0 \leq \ell \leq L$
Representation	Assigns one model-dependent uncertainty degree to each element.	Decomposes each uncertainty degree into multiple interpretable subcomponents.	Applies refinement repeatedly across several compatible levels.
Structure	Single uncertainty level.	One refinement level.	Hierarchical sequence of refinement levels.
Compatibility	Not required.	$\mu_A = \pi_0 \circ \tilde{\mu}_A$.	$\mu_A^{(\ell)} = \pi_\ell \circ \mu_A^{(\ell+1)}$.
Main purpose	Unified representation of uncertainty models.	More detailed representation of heterogeneous evidence.	Multilevel and hierarchical representation of uncertainty.
Special-case relation	Base model.	Generalizes an Uncertain Set.	Generalizes both preceding models.

4.2 Multidimensional Iterated Refined Uncertain Sets

The ordinary notion of an Iterated Refined Uncertain Set uses a one-dimensional refinement index

$$\ell \in \{0, 1, \dots, L\}.$$

However, uncertainty may be refined simultaneously along several independent directions. For example, one refinement coordinate may represent the decomposition of evidence sources, while another may represent semantic categories, temporal scales, expert groups, spatial resolutions, or priority levels. This motivates the following multidimensional extension.



Definition 4.10 (Multidimensional Refinement Index). Let $m \geq 1$ be an integer and let

$$\mathbf{L} = (L_1, L_2, \dots, L_m) \in \mathbb{N}_{\geq 0}^m.$$

Define the finite multidimensional refinement index set by

$$\Lambda_{\mathbf{L}} = \prod_{i=1}^m \{0, 1, \dots, L_i\}.$$

An element of $\Lambda_{\mathbf{L}}$ is denoted by

$$\ell = (\ell_1, \ell_2, \dots, \ell_m).$$

The set $\Lambda_{\mathbf{L}}$ is equipped with the coordinatewise partial order

$$\ell \preceq \mathbf{r} \iff \ell_i \leq r_i \text{ for every } i \in \{1, \dots, m\}.$$

The zero index and the maximal refinement index are denoted by

$$\mathbf{0} = (0, \dots, 0) \quad \text{and} \quad \mathbf{L} = (L_1, \dots, L_m),$$

respectively.

Definition 4.11 (Multidimensional System of Refined Degree Domains). Let $\mathfrak{M}$ be an uncertainty model with degree domain

$$D^{(\mathbf{0})} = \text{Dom}(\mathfrak{M}).$$

For every multidimensional refinement index

$$\ell = (\ell_1, \dots, \ell_m) \in \Lambda_{\mathbf{L}},$$

let

$$D^{(\ell)} = D^{(\ell_1, \dots, \ell_m)}$$

be an uncertainty degree domain at refinement level ℓ .

For every pair

$$\ell \preceq \mathbf{r},$$

let

$$\pi_{\ell, \mathbf{r}} : D^{(\mathbf{r})} \longrightarrow D^{(\ell)}$$

be an aggregation or coarsening map. These maps are required to satisfy:

(i) For every $\ell \in \Lambda_{\mathbf{L}}$,

$$\pi_{\ell, \ell} = \text{id}_{D^{(\ell)}}.$$

(ii) For all

$$\ell \preceq \mathbf{r} \preceq \mathbf{s},$$

one has

$$\pi_{\ell, \mathbf{s}} = \pi_{\ell, \mathbf{r}} \circ \pi_{\mathbf{r}, \mathbf{s}}.$$

The family

$$\mathcal{D}_{\mathbf{L}} = \left(\left\{ D^{(\ell)} \right\}_{\ell \in \Lambda_{\mathbf{L}}}, \left\{ \pi_{\ell, \mathbf{r}} \right\}_{\ell \preceq \mathbf{r}} \right)$$

is called a *multidimensional system of refined uncertainty degree domains*.

Definition 4.12 (Multidimensional Iterated Refined Uncertain Set). Let X be a nonempty universe, let $\mathfrak{M}$ be an uncertainty model, and let

$$\mathcal{D}_{\mathbf{L}} = \left(\left\{ D^{(\ell)} \right\}_{\ell \in \Lambda_{\mathbf{L}}}, \left\{ \pi_{\ell, \mathbf{r}} \right\}_{\ell \preceq \mathbf{r}} \right)$$

be a multidimensional system of refined uncertainty degree domains.

An *m-dimensional Iterated Refined Uncertain Set of depth $\mathbf{L}$* on X , abbreviated as an *m-DIRU-Set*, is a compatible family of mappings

$$\mu_A^{(\ell)} : X \longrightarrow D^{(\ell)}, \quad \ell \in \Lambda_{\mathbf{L}},$$

such that, whenever

$$\ell \preceq \mathbf{r},$$

the compatibility condition

$$\mu_A^{(\ell)} = \pi_{\ell, \mathbf{r}} \circ \mu_A^{(\mathbf{r})}$$

holds.

Equivalently, for every $x \in X$,

$$\mu_A^{(\ell)}(x) = \pi_{\ell, \mathbf{r}} \left(\mu_A^{(\mathbf{r})}(x) \right).$$

The complete multidimensional structure may be written as

$$\mathcal{A}_{\mathbf{L}} = \left(X, \left\{ D^{(\ell)} \right\}_{\ell \in \Lambda_{\mathbf{L}}}, \left\{ \pi_{\ell, \mathbf{r}} \right\}_{\ell \preceq \mathbf{r}}, \left\{ \mu_A^{(\ell)} \right\}_{\ell \in \Lambda_{\mathbf{L}}} \right).$$



Remark 4.13 (Finest Multidimensional Degree). Because the finite index set $\Lambda_{\mathbf{L}}$ has the maximal element

$$\mathbf{L} = (L_1, \dots, L_m),$$

the finest uncertainty degree of an element $x \in X$ is

$$\mu_A^{(\mathbf{L})}(x) \in D^{(\mathbf{L})} = D^{(L_1, \dots, L_m)}.$$

Every coarser degree is obtained from this finest degree by

$$\mu_A^{(\ell)} = \pi_{\ell, \mathbf{L}} \circ \mu_A^{(\mathbf{L})}.$$

In particular, the ordinary uncertainty degree is

$$\mu_A^{(0)} = \pi_{0, \mathbf{L}} \circ \mu_A^{(\mathbf{L})}.$$

Thus,

$$X \xrightarrow{\mu_A^{(\mathbf{L})}} D^{(L_1, \dots, L_m)} \xrightarrow{\pi_{0, \mathbf{L}}} D^{(0)} = \text{Dom}(\mathfrak{M})$$

recovers the underlying Uncertain Set.

Definition 4.14 (Elementary Directional Coarsening Maps). For each $i \in \{1, \dots, m\}$, let

$$\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$$

denote the i -th standard basis vector of $\mathbb{N}^m$.

Whenever

$$\ell, \ell + \mathbf{e}_i \in \Lambda_{\mathbf{L}},$$

define an elementary coarsening map in the i -th refinement direction by

$$\rho_i^{(\ell)} : D^{(\ell + \mathbf{e}_i)} \longrightarrow D^{(\ell)}.$$

These elementary maps are said to be *directionally compatible* if, for every $i \neq j$,

$$\rho_i^{(\ell)} \circ \rho_j^{(\ell + \mathbf{e}_i)} = \rho_j^{(\ell)} \circ \rho_i^{(\ell + \mathbf{e}_j)}$$

whenever all the degree domains appearing in this equality are defined.

Proposition 4.15 (Directional Compatibility Implies Path Independence). Suppose that the elementary directional coarsening maps

$$\rho_i^{(\ell)} : D^{(\ell + \mathbf{e}_i)} \longrightarrow D^{(\ell)}$$

satisfy

$$\rho_i^{(\ell)} \circ \rho_j^{(\ell + \mathbf{e}_i)} = \rho_j^{(\ell)} \circ \rho_i^{(\ell + \mathbf{e}_j)}$$

for every $i \neq j$ whenever the corresponding degree domains are defined.

Then, for every

$$\ell \preceq \mathbf{r},$$

the composite coarsening map

$$\pi_{\ell, \mathbf{r}} : D^{(\mathbf{r})} \longrightarrow D^{(\ell)}$$

obtained by composing elementary directional coarsening maps is independent of the chosen monotone coarsening path from $\mathbf{r}$ to ℓ .

Proof. A monotone coarsening path from $\mathbf{r}$ to ℓ consists of

$$\mathbf{r}_i - \ell_i$$

elementary coarsening steps in direction i , for each $i = 1, \dots, m$.

Hence every such path contains exactly the same multiset of directional steps; only their order may differ. Any two orderings of this finite multiset can be transformed into one another by a finite sequence of transpositions of adjacent steps in distinct directions.

Consider one such adjacent interchange, involving directions i and j . The two corresponding local composites are

$$\rho_i^{(\ell)} \circ \rho_j^{(\ell + \mathbf{e}_i)}$$

and

$$\rho_j^{(\ell)} \circ \rho_i^{(\ell + \mathbf{e}_j)}.$$

By directional compatibility, these maps are equal. Therefore exchanging two adjacent directional steps does not change the resulting composite coarsening map.

Since any two monotone paths are connected by finitely many such adjacent exchanges, all monotone paths from $\mathbf{r}$ to ℓ induce the same composite. Thus

$$\pi_{\ell, \mathbf{r}}$$

is well defined independently of the chosen path. □



Example 4.16 (Failure of Path Independence). Consider the two-dimensional case

$$m = 2, \quad \mathbf{L} = (1, 1).$$

Let

$$D^{(0,0)} = D^{(1,0)} = D^{(0,1)} = [0, 1], \quad D^{(1,1)} = [0, 1]^2.$$

Define the elementary directional coarsening maps by

$$\rho_2^{(1,0)} : D^{(1,1)} \longrightarrow D^{(1,0)}, \quad \rho_2^{(1,0)}(x, y) = x,$$

and

$$\rho_1^{(0,1)} : D^{(1,1)} \longrightarrow D^{(0,1)}, \quad \rho_1^{(0,1)}(x, y) = y.$$

At the lower level, let

$$\rho_1^{(0,0)}(z) = z, \quad \rho_2^{(0,0)}(z) = z.$$

There are two monotone coarsening paths from $D^{(1,1)}$ to $D^{(0,0)}$. Along the first path,

$$D^{(1,1)} \xrightarrow{\rho_2^{(1,0)}} D^{(1,0)} \xrightarrow{\rho_1^{(0,0)}} D^{(0,0)},$$

we obtain

$$(\rho_1^{(0,0)} \circ \rho_2^{(1,0)})(x, y) = x.$$

Along the second path,

$$D^{(1,1)} \xrightarrow{\rho_1^{(0,1)}} D^{(0,1)} \xrightarrow{\rho_2^{(0,0)}} D^{(0,0)},$$

we obtain

$$(\rho_2^{(0,0)} \circ \rho_1^{(0,1)})(x, y) = y.$$

For example, for

$$d = (0.8, 0.2) \in D^{(1,1)},$$

the first path gives

$$0.8,$$

whereas the second path gives

$$0.2.$$

Therefore,

$$\rho_1^{(0,0)} \circ \rho_2^{(1,0)} \neq \rho_2^{(0,0)} \circ \rho_1^{(0,1)}.$$

Hence the elementary square does not commute, and there is no path-independent coarsening map

$$\pi_{(0,0),(1,1)} : D^{(1,1)} \longrightarrow D^{(0,0)}$$

determined by these elementary maps.

This example shows that directional compatibility is not merely a technical convenience: without it, the final coarse uncertainty degree may depend on the order in which different refinement directions are aggregated.

Example 4.17 (Two-Dimensional Iterated Refinement). Let $m = 2$, and suppose that the two refinement coordinates represent:

- (i) refinement by evidence source;
- (ii) refinement by semantic uncertainty type.

A degree domain at level

$$(\ell_1, \ell_2)$$

is written as

$$D^{(\ell_1, \ell_2)}.$$

For example,

$$D^{(0,0)}$$

is the ordinary uncertainty degree domain,

$$D^{(1,0)}$$

refines the evidence-source structure,

$$D^{(0,1)}$$

refines the semantic uncertainty structure, and

$$D^{(1,1)}$$

simultaneously refines both structures.

The corresponding membership mappings are

$$\mu_A^{(0,0)} : X \longrightarrow D^{(0,0)},$$

$$\mu_A^{(1,0)} : X \longrightarrow D^{(1,0)}, \quad \mu_A^{(0,1)} : X \longrightarrow D^{(0,1)},$$

and

$$\mu_A^{(1,1)} : X \longrightarrow D^{(1,1)}.$$



Their compatibility is expressed by

$$\mu_A^{(0,0)} = \pi_{(0,0),(1,0)} \circ \mu_A^{(1,0)} = \pi_{(0,0),(0,1)} \circ \mu_A^{(0,1)}$$

and

$$\begin{aligned}\mu_A^{(1,0)} &= \pi_{(1,0),(1,1)} \circ \mu_A^{(1,1)}, \\ \mu_A^{(0,1)} &= \pi_{(0,1),(1,1)} \circ \mu_A^{(1,1)}.\end{aligned}$$

Hence,

$$\pi_{(0,0),(1,0)} \circ \pi_{(1,0),(1,1)} = \pi_{(0,0),(0,1)} \circ \pi_{(0,1),(1,1)}.$$

Thus, the ordinary degree obtained from $D^{(1,1)}$ is independent of the order in which the two refinement directions are aggregated.

Proposition 4.18 (Reduction to Earlier Models). *Let $\mathcal{A}_L$ be an m -dimensional Iterated Refined Uncertain Set.*

(i) *If $m = 1$, then $\mathcal{A}_L$ reduces to an ordinary Iterated Refined Uncertain Set.*

(ii) *If*

$$L_1 = \cdots = L_m = 0,$$

then $\mathcal{A}_L$ reduces to an ordinary Uncertain Set.

(iii) *If only one coordinate L_i is positive and all other coordinates are zero, then the structure reduces to a one-dimensional Iterated Refined Uncertain Set along the i -th refinement direction.*

(iv) *For every fixed choice of*

$$(\ell_2, \dots, \ell_m),$$

the family

$$\left\{ D^{(\ell_1, \ell_2, \dots, \ell_m)} \right\}_{\ell_1=0}^{L_1}$$

forms a one-dimensional refinement chain in the first coordinate. Analogous statements hold for every other coordinate.

Proof. Part (i) follows because the coordinatewise partial order on Λ_L becomes the ordinary linear order

$$0 \leq 1 \leq \cdots \leq L_1$$

when $m = 1$.

Part (ii) follows because

$$\Lambda_L = \{\mathbf{0}\},$$

so the only membership mapping is

$$\mu_A^{(\mathbf{0})} : X \longrightarrow \text{Dom}(\mathfrak{M}).$$

Part (iii) is immediate because the index set is effectively a one-dimensional chain along the only nonzero coordinate.

Part (iv) follows by fixing all coordinates except one and restricting the multidimensional inverse system to the corresponding linearly ordered subset of Λ_L . $\square$

Definition 4.19 (Infinitely Multidimensional Iterated Refined Uncertain Set). Let the refinement index set be

$$\Lambda_\infty = \mathbb{N}_{\geq 0}^m$$

with the coordinatewise partial order.

For every

$$\ell \in \Lambda_\infty,$$

let $D^{(\ell)}$ be a degree domain, and for every

$$\ell \preceq \mathbf{r},$$

let

$$\pi_{\ell, \mathbf{r}} : D^{(\mathbf{r})} \longrightarrow D^{(\ell)}$$

satisfy

$$\pi_{\ell, \ell} = \text{id}_{D^{(\ell)}}$$

and

$$\pi_{\ell, \mathbf{s}} = \pi_{\ell, \mathbf{r}} \circ \pi_{\mathbf{r}, \mathbf{s}}$$

whenever

$$\ell \preceq \mathbf{r} \preceq \mathbf{s}.$$

An *Infinitely Multidimensional Iterated Refined Uncertain Set* on X is a family

$$\left\{ \mu_A^{(\ell)} : X \longrightarrow D^{(\ell)} \right\}_{\ell \in \mathbb{N}_{\geq 0}^m}$$

such that

$$\mu_A^{(\ell)} = \pi_{\ell, \mathbf{r}} \circ \mu_A^{(\mathbf{r})}$$

for every

$$\ell \preceq \mathbf{r}.$$



Remark 4.20 (Inverse-Limit Representation). The multidimensional inverse limit is

$$\varprojlim_{\ell \in \mathbb{N}_{\geq 0}^m} D^{(\ell)} = \left\{ (d_\ell)_\ell \in \prod_{\ell \in \mathbb{N}_{\geq 0}^m} D^{(\ell)} \mid \pi_{\ell, \mathbf{r}}(d_{\mathbf{r}}) = d_\ell \text{ whenever } \ell \preceq \mathbf{r} \right\}.$$

Therefore, an Infinitely Multidimensional Iterated Refined Uncertain Set may equivalently be represented by a mapping

$$\mu_A^{(\infty)} : X \longrightarrow \varprojlim_{\ell \in \mathbb{N}_{\geq 0}^m} D^{(\ell)}.$$

Remark 4.21 (Interpretation of the Refinement Coordinates). The m refinement coordinates may represent independent aspects of uncertainty. For example, one may interpret

$$D^{(\ell_1, \ell_2, \ell_3, \ell_4)}$$

as a degree domain refined according to:

- (i) ℓ_1 : evidence-source refinement;
- (ii) ℓ_2 : semantic refinement;
- (iii) ℓ_3 : temporal refinement;
- (iv) ℓ_4 : spatial or hierarchical refinement.

Hence, the multidimensional framework distinguishes the *depth of refinement in each independent direction*, rather than describing refinement by only one total level.

For reference, a comparison of Iterated Refined Uncertain Sets and Multidimensional Iterated Refined Uncertain Sets is presented in Table 7.

Table 7: Comparison of Iterated Refined Uncertain Sets and Multidimensional Iterated Refined Uncertain Sets

Aspect	Iterated Refined Uncertain Set	Multidimensional Iterated Refined Uncertain Set
Refinement index	A single level $\ell \in \{0, 1, \dots, L\}.$	A multi-index $\ell = (\ell_1, \dots, \ell_m).$
Degree domain	$D^{(\ell)}.$	$D^{(\ell_1, \dots, \ell_m)}.$
Index structure	A linearly ordered refinement chain.	A coordinatewise partially ordered refinement system.
Refinement directions	One refinement direction.	Several independent refinement directions.
Membership mappings	$\mu_A^{(\ell)} : X \rightarrow D^{(\ell)}.$	$\mu_A^{(\ell)} : X \rightarrow D^{(\ell)}.$
Compatibility	$\mu_A^{(\ell)} = \pi_\ell \circ \mu_A^{(\ell+1)}.$	$\mu_A^{(\ell)} = \pi_{\ell, \mathbf{r}} \circ \mu_A^{(\mathbf{r})}, \quad \ell \preceq \mathbf{r}.$
Additional condition	Successive refinements must be compatible.	Directional coarsening maps must satisfy commutativity or path-independence.
Main purpose	Represents uncertainty through successive hierarchical refinement.	Represents uncertainty refined independently along several dimensions.
Special-case relation	The basic one-dimensional iterative model.	Reduces to the one-dimensional model when $m = 1$.

5 Conclusion

In this paper, we introduced an n -Refined Neutrosophic Logic (n-RNL) that decomposes the classical truth-indeterminacy-falsity triplet into interpretable subcomponents. Within this framework we instantiated refined quadripartitioned, pentapartitioned, and heptapartitioned families that explicitly separate *contradiction*, *ignorance*, *unknown*, and *relative* (context-dependent) truth/falsity. Logical connectives are defined blockwise by a chosen t -norm / t -conorm pair, preserving associativity, commutativity, and monotonicity componentwise. We proved merge/split homomorphisms that (i) reduce refined models to their unrefined counterparts without breaking



algebraic structure, and (ii) embed unrefined semantics faithfully into refined spaces. We further proposed *Iterative Refined Neutrosophic Logic* (IRNL), which supports repeated, symmetric refinements and guarantees functorial stability of evaluations: refining then evaluating yields the same result as evaluating then refining.

In addition, we extended the refinement principle beyond neutrosophic logic to the general framework of Uncertain Sets. We introduced Refined Uncertain Sets, finite-depth and infinitely Iterated Refined Uncertain Sets, and Multidimensional Iterated Refined Uncertain Sets indexed by coordinatewise partially ordered multi-indices. The associated aggregation maps provide compatible coarsening operations, while the path-independence condition ensures that multidirectional refinement yields a coherent uncertainty degree independently of the chosen coarsening path.

As future work, we plan to extend this program over richer combinatorial and set-theoretic substrates, including graphs [67], hypergraphs [68, 69], superhypergraphs [70, 71], plithogenic sets [72, 73], and Chemical Graphs [74]. We expect these directions to enable more expressive modeling of heterogeneous evidence, multi-source contradictions, and context-sensitive reasoning at scale.

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References

- [1] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.
- [2] Takaaki Fujita and Florentin Smarandache. *A Survey of HyperSoft Set Theory and Its Extensions*. Neutrosophic Science International Association (NSIA) Publishing House, 2026.
- [3] Florentin Smarandache. *New types of soft sets “hypersoft set, indeterminsoft set, indeterminhypersoft set, and treesoft set”: an improved version*. Infinite Study, 2023.
- [4] Haibin Wang, Florentin Smarandache, Yanqing Zhang, and Rajshekhar Sunderraman. *Single valued neutrosophic sets*. Infinite study, 2010.
- [5] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.
- [6] Takaaki Fujita and Florentin Smarandache. *A Revised, Comprehensive, and Re-Updated Survey of Unified Frameworks for Fuzzy, Neutrosophic, and Uncertain Sets*. Neutrosophic Science International Association (NSIA) Publishing House, 2026.
- [7] G Muhiuddin, Satham Hussain S, and Durga Nagarajan. Quadripartitioned bipolar neutrosophic competition graph with novel application. *Neutrosophic Sets and Systems*, 82(1):8, 2025.
- [8] Muhammad Akram, Musavarah Sarwar, Wieslaw A Dudek, Muhammad Akram, Musavarah Sarwar, and Wieslaw A Dudek. Bipolar neutrosophic graph structures. *Graphs for the Analysis of Bipolar Fuzzy Information*, pages 393–446, 2021.
- [9] Takaaki Fujita and Florentin Smarandache. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2024.
- [10] K Settu and M Jayalakshmi. Prediction of risk factor in hepatitis diagnosis using interval value hyperneutrosophic and einstein aggregated operations with extended mcdm. *Expert Systems with Applications*, page 130364, 2025.
- [11] Hu Zhao and Hong-Ying Zhang. On hesitant neutrosophic rough set over two universes and its application. *Artificial Intelligence Review*, 53:4387–4406, 2020.
- [12] Qaisar Khan, Peide Liu, and Tahir Mahmood. Some generalized dice measures for double-valued neutrosophic sets and their applications. *Mathematics*, 6(7):121, 2018.
- [13] Takaaki Fujita. Triple-valued neutrosophic set, quadruple-valued neutrosophic set, quintuple-valued neutrosophic set, and double-valued indeterminsoft set. *Neutrosophic Systems with Applications*, 25(5):3, 2025.
- [14] Sumbal Ali, Asad Ali, Ahmad Bin Azim, Ahmad Aloqaily, and Nabil Mlaiki. Utilizing aggregation operators based on q-rung orthopair neutrosophic soft sets and their applications in multi-attributes decision making problems. *Heliyon*, 2024.
- [15] S Santhoshkumar, J Aldring, and D Ajay. Analyzing aggregation operators on complex q-rung orthopair neutrosophic sets with their application. In *International Conference on Intelligent and Fuzzy Systems*, pages 744–751. Springer, 2024.
- [16] Michael Gr Voskoglou, Florentin Smarandache, and Mona Mohamed. *q-Rung Neutrosophic Sets and Topological Spaces*. Infinite Study, 2024.
- [17] John N Mordeson and Premchand S Nair. *Fuzzy graphs and fuzzy hypergraphs*, volume 46. Physica, 2012.
- [18] Krassimir T Atanassov and G Gargov. *Intuitionistic fuzzy logics*. Springer, 2017.
- [19] Sujit Das and Samarjit Kar. Intuitionistic multi fuzzy soft set and its application in decision making. In *Pattern Recognition and Machine Intelligence: 5th International Conference, PReMI 2013, Kolkata, India, December 10-14, 2013. Proceedings 5*, pages 587–592. Springer, 2013.
- [20] Qinghua Zhang, Jin Wang, Guoyin Wang, and Hong Yu. The approximation set of a vague set in rough approximation space. *Information Sciences*, 300:1–19, 2015.
- [21] An Lu and Wilfred Ng. Vague sets or intuitionistic fuzzy sets for handling vague data: which one is better? In *International conference on conceptual modeling*, pages 401–416. Springer, 2005.
- [22] Florentin Smarandache. *Indeterminacy Fuzzy Set coincides with the Picture Fuzzy Set, and both are particular cases of the Refined Neutrosophic Set*. Infinite Study, 2026.
- [23] Bui Cong Cuong and Vladik Kreinovich. Picture fuzzy sets-a new concept for computational intelligence problems. In *2013 third world congress on information and communication technologies (WICT 2013)*, pages 1–6. IEEE, 2013.
- [24] Takaaki Fujita. Operations, similarity measures, and rough-set approximations for uncertain sets. *Vietnam Journal of Science, Technology and Engineering*, 2026.
- [25] Takaaki Fujita and Florentin Smarandache. *A Comprehensive and Updated Survey of Fuzzy, Neutrosophic, and Uncertain Sets*. Neutrosophic Science International Association (NSIA) Publishing House, 2026.
- [26] Vicenç Torra and Yasuo Narukawa. On hesitant fuzzy sets and decision. In *2009 IEEE international conference on fuzzy systems*, pages 1378–1382. IEEE, 2009.



- [27] Zeshui Xu. *Hesitant fuzzy sets theory*, volume 314. Springer, 2014.
- [28] S Satham Hussain, N Durga, Muhammad Aslam, G Muhiuddin, and Ganesh Ghorai. New concepts on quadripartitioned neutrosophic competition graph with application. *International Journal of Applied and Computational Mathematics*, 10(2):57, 2024.
- [29] Takaaki Fujita and Florentin Smarandache. Some types of hyperneutrosophic set (3): Dynamic, quadripartitioned, pentapartitioned, heptapartitioned, m-polar. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, 2025.
- [30] Shoaib Arbab, Muhammad Idrees, Aqeelat Hussain, Walid Abdelfattah, Arif Mehmood, and Cris L Armada. A tangent similarity-based pentapartitioned neutrosophic soft topological framework for cancer diagnosis. *International Journal of Analysis and Applications*, 24:270–270, 2026.
- [31] Tanmay Chatterjee, Subhabrata Mondal, and Avishek Chakraborty. Selection of customer relations manager for logistics hub using triangular fuzzy penta-partitioned neutrosophic einstein’s aggregation operator in triangular penta-partitioned neutrosophic arena. *Quality & Quantity*, pages 1–59, 2026.
- [32] Takaaki Fujita. Advanced partitioned neutrosophic sets: Formalization of hexa-, hepta-, octa-, nona-, and deca-partitioned structures. *Abhath Journal of Basic and Applied Sciences*, 4(2):40–60, 2025.
- [33] Takaaki Fujita, R. Hatamleh, and A. Heilat. Advanced partitioned neutrosophic offsets, oversets, and undersets: Modeling “neither agree nor disagree” and beyond. *Statistics, Optimization & Information Computing*, 2026.
- [34] Surapati Pramanik. Interval quadripartitioned neutrosophic sets. *Neutrosophic Sets and Systems*, vol. 51/2022: *An International Journal in Information Science and Engineering*, page 146, 2022.
- [35] Mithun Datta, Himangshu Nath, and Kalyani Debnath. Pentapartitioned neutrosophic quasi coincident topological space. *New Trends in Neutrosophic Theories and Applications, Volume V*, page 47, 2026.
- [36] Suman Das, Prasanna Poojary, Vadiraja Bhatta GR, and Sunny Kumar Sharma. Single-valued pentapartitioned neutrosophic bi-topological spaces. *International Journal of Neutrosophic Science (IJNS)*, 23(4), 2024.
- [37] S Broumi and Tomasz Witczak. Heptapartitioned neutrosophic soft set. *International Journal of Neutrosophic Science*, 18(4):270–290, 2022.
- [38] Maha Mohammed Saeed, Raed Hatamleh, Hamza Ali Abujabal, Yahya Khan, Abdallah Al-Husban, Amy A Laja, Sulfaia MM Pangilan, Cris L Armada, Jamil J Hamja, and Arif Mehmood Khattak. Heptapartitioned neutrosophic soft topologies and machine learning techniques for exploring romantic feelings. *European Journal of Pure and Applied Mathematics*, 18(3):6221–6221, 2025.
- [39] Florentin Smarandache. n-valued refined neutrosophic logic and its applications to physics. *Infinite study*, 4:143–146, 2013.
- [40] Atharva Rajesh Shekatkar, Aditya Deodeshmukh, and Ilanthenral Kandasamy. Refined neutrosophic cognitive maps: A novel framework for modeling indeterminate causal relationships. *International Journal of Fuzzy Systems*, pages 1–24, 2025.
- [41] Min Zhang, Yaozu Wang, Rui Huang, Qiao Wu, and Bingqing Li. A novel neutrosophic multi-layered complex refined hyperstructure: Theoretical advancement with illustrative examples from employee mental health education effectiveness in enterprises. *Neutrosophic Sets and Systems*, 87(1):11, 2025.
- [42] Ankita Ankita and Deepak Kumar. Reliability assessment of refined single-valued neutrosophic data with application in multi-criteria decision-making. *Neutrosophic Sets and Systems*, 94(1):6, 2025.
- [43] Ilanthenral Kandasamy and Florentin Smarandache. Triple refined indeterminate neutrosophic sets for personality. *Collected Papers. Volume VII: On Neutrosophic Theory and Applications*, page 86, 2022.
- [44] Salman A Khan, Kashif Iqbal, Nazeeruddin Mohammad, Rehan Akbar, Syed Saad Azhar Ali, and Ammar Ahmed Siddiqui. A novel fuzzy-logic-based multi-criteria metric for performance evaluation of spam email detection algorithms. *Applied Sciences*, 12(14):7043, 2022.
- [45] Oryan Henig and Keith S Kaye. Bacterial pneumonia in older adults. *Infectious disease clinics of North America*, 31(4):689, 2017.
- [46] Mohammad Abobala and Muritala Ibrahim. An introduction to refined neutrosophic number theory. *Neutrosophic sets and systems*, 45:40–53, 2021.
- [47] Ying Zheng, Lie Mo, and Zhiwen Ye. A robust multi-dimensional n-valued refined neutrosophic logic framework for competitive calculation in the leisure sports industry. *Neutrosophic Sets and Systems*, 86:631–643, 2025.
- [48] André S Fialho, Susana M Vieira, Uzay Kaymak, Rui J Almeida, Federico Cismonti, Shane R Reti, Stan N Finkelstein, and João MC Sousa. Mortality prediction of septic shock patients using probabilistic fuzzy systems. *Applied Soft Computing*, 42:194–203, 2016.
- [49] Hudson Henrique Gomes Pires, Fábio Fernandes Neves, and Antonio Pazin-Filho. Triage and flow management in sepsis. *International Journal of Emergency Medicine*, 12(1):36, 2019.
- [50] P Yiarayong. Some weighted aggregation operators of quadripartitioned single-valued trapezoidal neutrosophic sets and their multi-criteria group decision-making method for developing green supplier selection criteria. *OPSEARCH*, pages 1–55, 2024.
- [51] Bimal Shil, Rakhal Das, Carlos Granados, Suman Das, and Binod Chandra Tripathy. Single-valued quadripartitioned neutrosophic d-ideal of d-algebra. *Neutrosophic Sets and Systems*, 67:105–114, 2024.
- [52] Satham S Hussain, Muhammad Aslam, Hossein Rahmonlou, and N Durga. Applying interval quadripartitioned single-valued neutrosophic sets to graphs and climatic analysis. In *Data-Driven Modelling with Fuzzy Sets*, pages 100–143. CRC Press, 2025.
- [53] Tanmoy Chatterjee and Surapati Pramanik. Triangular fuzzy quadripartitioned neutrosophic set and its properties. *Neutrosophic Sets and Systems*, 75:15–28, 2025.
- [54] Rakhal Das and Suman Das. Pentapartitioned neutrosophic subtraction algebra. *Neutrosophic Sets and Systems*, 68:89–98, 2024.
- [55] M Myvizhi, Mohamed Elkholy, Ahmed Abdelhafeez, Hussam Elbehiery, et al. Enhanced madm strategy with heptapartitioned neutrosophic distance metrics. *Neutrosophic Sets and Systems*, 78:74–96, 2025.
- [56] T Mythili, V Jeyanthi, et al. Enhanced decision-making academic libraries with topsis methods using blockchain technology by heptapartitioned neutrosophic number. In *Leveraging Blockchain for Future-Ready Libraries*, pages 123–140. IGI Global Scientific Publishing, 2025.
- [57] Hussam Elbehiery. Enhanced madm strategy with heptapartitioned neutrosophic distance metrics. *Neutrosophic Sets and Systems*, vol. 78/2025: *An International Journal in Information Science and Engineering*, page 74, 2025.
- [58] Shera Tonazzi, Leanne Prenovost, and Suzette Scheuermann. Delayed antibiotic prescribing to reduce antibiotic use: an urgent care practice change. *BMJ Open Quality*, 11(1), 2022.
- [59] Heather Marie MacNeil and Bonnie Mak. Constructions of authenticity. *Library Trends*, 56(1):26–52, 2007.
- [60] Shan Ye and Jun Ye. Dice similarity measure between single valued neutrosophic multisets and its application in medical diagnosis. *Neutrosophic sets and systems*, 6(1):9, 2014.



- [61] TK Shinoj and Sunil Jacob John. Intuitionistic fuzzy multisets and its application in medical diagnosis. *World academy of science, engineering and technology*, 6(1):1418–1421, 2012.
- [62] André L Cristiani, Douglas D Lieira, Rodolfo I Meneguette, and Heloisa A Camargo. A fuzzy intrusion detection system for identifying cyber-attacks on iot networks. In *2020 IEEE Latin-American Conference on Communications (LATINCOM)*, pages 1–6. Ieee, 2020.
- [63] Yoosef B Abushark, A Irshad Khan, Fawaz Alsolami, Abdulmohsen Almalawi, M Mottahir Alam, Alka Agrawal, Rajeev Kumar, and Raees Ahmad Khan. Cyber security analysis and evaluation for intrusion detection systems. *Comput. Mater. Contin.*, 72(1):1765–1783, 2022.
- [64] Takaaki Fujita and Florentin Smarandache. Fuzzy, neutrosophic, and uncertain graph theory: Properties and applications. *arXiv e-prints*, pages arXiv–2605, 2026.
- [65] Takaaki Fujita and Florentin Smarandache. *A Survey of Fuzzy and Uncertain Concepts in Applied Mathematics*. Infinite Study, 2025.
- [66] Takaaki Fujita and Florentin Smarandache. *A comprehensive survey of set-theoretic concepts related to fuzzy, neutrosophic, and uncertain sets*. Infinite Study, 2026.
- [67] Takaaki Fujita, Ajoy Kanti Das, and Suman Das. Meta-graphs and iterated meta-graphs: A novel framework for hierarchical network analysis and complex system modeling. *Applied Research Advances*, pages 1–12, 2027.
- [68] Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering*. Cham: Springer, 1, 2013.
- [69] Yifan Feng, Haoxuan You, Zizhao Zhang, Rongrong Ji, and Yue Gao. Hypergraph neural networks. In *Proceedings of the AAAI conference on artificial intelligence*, volume 33, pages 3558–3565, 2019.
- [70] Takaaki Fujita and Florentin Smarandache. *Representing higher-order networks: A survey of graph-based frameworks*. Infinite Study, 2026.
- [71] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.
- [72] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.
- [73] Prem Kumar Singh et al. Dark data analysis using intuitionistic plithogenic graphs. *International Journal of Neutrosophic Sciences*, 16(2):80–100, 2021.
- [74] Ramón García-Domenech, Jorge Gálvez, Jesus V de Julián-Ortiz, and Lionello Pogliani. Some new trends in chemical graph theory. *Chemical Reviews*, 108(3):1127–1169, 2008.

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