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A new generalization of exponentiated exponential distribution using quantile functions

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ABSTRACT

Generalising existing probability distributions increases their appeal to researchers and expands their applicability to real-life situations by adding flexibility to the existing models. In this study, a new generalised class of the exponentiated exponential (EE) distribution is introduced, referred to as the T-Exponential Exponential{Y} or T-EE{Y} class of distributions. By utilising the quantile functions of several well-known continuous distributions in the T-EE{Y} framework, six distinct subclasses have been developed. Various statistical properties such as quantiles, mode, incomplete moments, entropy, and mean deviation have been derived for these subclasses. Additionally, four specific member models within the proposed class have been explored. The study demonstrates that the members from the T-EE{Y} class exhibit flexible shapes, including unimodal, bimodal, symmetrical, and skewed (both right and left) forms. Parameter estimation is performed using the maximum likelihood estimation method, and the effectiveness of the estimators is evaluated through a Monte Carlo simulation study. To evaluate the practical applicability of the proposed class of distributions, three real-world datasets are analysed. The members of the proposed class consistently outperform several existing distributions in modelling lifetime data, showcasing its significance and versatility.

Keywords: exponentiated; T-EE{Y}, EED; generalization; quantiles; entropy

1. Introduction

The concept of exponentiated distributions was first introduced by [1], who developed a new family of distributions known as Exponentiated Exponential distribution, having two parameters (shape and slope). It is noted that when dealing with real lifetime data sets then the generalised exponential (exponentiated exponential) distribution, having two parameters, shape and scale, performs better than the Gamma and Weibull distribution. The hazard rate of the exponentiated exponential distribution is very attractive and has different shapes upon the shape of the parameter. Many researchers have started to apply the exponentiated method to different traditional distributions. [2] have studied some exponentiated distributions. [3] derived various statistical properties of the exponentiated exponential distribution along with the sum, product, and ratio of the exponentiated exponential random variables. [4] introduced some extension in the exponentiated exponential distribution by using the Marshall Olkin technique, named as extended exponentiated exponential distribution. [5] proposed exponentiated moment exponential distribution with its various properties, applications, and conditional based characterisations as well. [6] proposed exponentiated exponential Weibull (EEWD) distribution. They used the Transformed-transformer T-X family distribution with different weights and studied the statistical properties of the distribution.

Generalisation of existing distributions provides greater flexibility in modelling real data sets. Different methodologies have been used to introduce more flexibility in distribution, mainly the addition of some parameters. Generalised families of distributions have been obtained by using different methodologies. A lot of distributions have been generalised to obtain generalised classes of distribution in recent decades and utilised to explain different real-life situations. In this article, we use the framework introduced by [7] as the T-R{Y}class of distribution to generalise two-parameter exponentiated exponential distributions and name it as T-EE{Y} framework. The idea behind the T-R{Y} class of distribution is as:

Let $W(F(x)) \vdots (0,1) \rightarrow (a,b)$ for $-\infty < (a,b) < +\infty,\ also\ a < b$. Suppose W be a monotonic and continuous function such that $\lim_{\delta \to 0^+} W(\delta) = a$ and $\lim_{\delta \to 1^-} W(\delta) = b$, then the function $G(x) = R\{w(F(x))\},\ \ -\infty < x < +\infty$ will be a distribution function if it meets the conditions of a distribution function.

i. $G(x)$ is increasing function

ii. $G(x)$ is a right continuous function

iii. $G(x) \rightarrow 0\ as\ x \rightarrow -\infty\ \ , G(x) \rightarrow 1\ as\ x \rightarrow +\infty$

The T-R{Y} framework is described as follows: Let T, R, and Y denote three continuous random variables with cumulative distribution functions. $F_T(x) = P(T \le x), F_R(x) = P(R \le x)$ and $F_Y(x) = P(Y \le x)$, respectively. The probability density functions of the random variable are $f_T(x)$, $f_R(x)$ and $f_R(x)$ and their quantile functions are $Q_T(p)$, $Q_R(p)$ and $Q_Y(p)$ respectively, where the quantile function is defined a as $Q_Y(p) = inf\ \{y\colon F_y(Y) \ge p\}$, $0 < p < 1$. Let a random variable (RV) "X" follow the T-R$\{$Y$\}$class of probability distributions, then the probability density function (pdf) and cumulative distribution function (cdf) are defined as,

$$F_X(x) = \int_a^{Q_Y(F_R(x))} f_T(t)dt \; = p(T \; \leq \; Q_Y(F_R(x)) \; = \; F_T(Q_Y(F_R(x))) \qquad (1)$$

$$f_X(x) = f_R(x) \, \frac{f_T(Q_Y(F_R(x)))}{f_T(Q_Y(F_R(x)))} \qquad (2)$$

The corresponding hazard function is given by

$$h_X(x) = h_R(x) \, \frac{h_T(Q_Y(F_R(x)))}{h_T(Q_Y(F_R(x)))} \qquad (3)$$

Where $h_R(x), h_X(x)\, , h_T(x)$ are the hazard functions of the random variables R, X, and T respectively. [8] proposed a new generator based on the beta distribution. [9] developed transformed-transformer, T-X family of distribution. [10] developed the exponentiated half logistic family of distribution, [11] studied the gamma-Weibull-G family of distribution, logistic-X by [12], an extension of Rayleigh distribution by [13] using T-X generator. [14] proposed a new generator based on the concept of the T-R$\{$Y$\}$ family of probability distributions. This generator gave the researcher a broad spectrum and many new classes of distribution were introduced. [15] Introduced a new generalisation of exponential distribution, which they called as T-exponential$\{$Y$\}$ class of distribution, [16] discussed new families of generalised Lomax distribution. [17] developed the family of Kumaraswamy-G- G (GK-G) family of distributions. [18] used the same technique to define the generalisation of the Pareto distribution. [19] used the T-R$\{$Y$\}$ framework to obtain the T-Lindley$\{$Y$\}$ class of distribution.

In this paper, the exponentiated exponential distribution (EED) is generalised using the T-R$\{$Y$\}$ framework. It has been observed that the EED performs better than the gamma and Weibull distributions when applied to real datasets. The EED's hazard rate function is particularly appealing due to its versatility in exhibiting different shapes. The primary motivation for adopting the "T-R$\{$Y$\}$ framework" is its ability to handle data with diverse shapes, including left/right skewed, symmetrical, unimodal and bimodal, as well as, which can be observed by the graphical representations of the selected members derived from the proposed class. Additionally, this approach seeks to offer models with fewer or more parameters, enabling better performance compared to prevailing probability distributions when modelling real-world phenomena.

The edifice of the rest of the article is as follows: Section 2 introduces the formulation of the "T-EE$\{$Y$\}$" family of probability models and proposes diverse sub-models. Section 3 derives statistical properties of the "T-EE$\{$Y$\}$ family", including the mode, moments, and entropies. Section

4 examines four sub-models of the "T-EE{Y} family" in detail. Section 5 describes the estimation of parameters for the proposed subclasses using the maximum likelihood method and includes a simulation study. Section 6 applies the new class of distributions to three real-world datasets, demonstrating its flexibility and effectiveness. Section 7 concludes with a summary and final remarks.

2. The T-exponentiated exponential {Y} class of probability models

Let an RV "R" be a continuous random variable which follows EED, then the cdf and pdf of the EED, respectively, are given as

$$F_R(x) = \left(1 - exp(-\lambda x)\right)^{\alpha} \;, \alpha > 0, \alpha > 0, \beta > 0 \qquad (4)$$

$$f_R(x) = \alpha\lambda exp(-\lambda x)\left(1 - exp(-\lambda x)\right)^{\alpha-1} \;, \alpha > 0, \alpha > 0, \beta > 0 \qquad (5)$$

Where α is the shape parameter and λ is the scale parameter.

Then, by using eq (1) with $F_R(x)$ from eq (4), and by using eq (2) with $f_R(x)$ from eq (5), the cdf of T-exponentiated exponentiated {Y} denoted as T-EE{Y}class of distributions are obtained as follows

$$F_X(x) = \int_a^{Q_y\left(\left(1-e^{-\lambda x}\right)^{\alpha}\right)} f_T(t)dt = F_T\left(Q_y\left(\left(1 - e^{-\lambda x}\right)^{\alpha}\right)\right) \qquad (6)$$

and the pdf of T-EE{Y} class can be expressed as

$$f_X(x) = \alpha\lambda e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\alpha-1} \frac{f_T\left(Q_y\left(\left(1-e^{-\lambda x}\right)^{\alpha}\right)\right)}{f_Y\left(Q_y\left(\left(1-e^{-\lambda x}\right)^{\alpha}\right)\right)} \qquad (7)$$

Using eq. (3), the corresponding hazard rate function (HRF) of T-EE{Y} class is

$$h_R(x) = \frac{\alpha\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1} f_T\left(Q_y\left(\left(1-e^{-\lambda x}\right)^{\alpha}\right)\right)}{\left\{1-F_T\left(Q_y\left(\left(1-e^{-\lambda x}\right)^{\alpha}\right)\right)\right\} f_Y\left(Q_y\left(\left(1-e^{-\lambda x}\right)^{\alpha}\right)\right)} \qquad (8)$$

2.1. T- EE {Y} sub probability models

In this section, six new sub-probability models of the "T-EE{Y} class" have been obtained by using the quantile functions (QF) given in Table 1.

Table 1:
Quantile functions for different choices of random variable Y along with the domain of random variable T

Distribution of Y	**Support of T**	**The quantile function $Q_Y(p)$**
Uniform	(0,1)	P
Exponential	(0, ∞)	$-\beta \log(1-P)\,, \quad \beta > 0$
Weibull	(0, ∞)	$-b[\log(1-P)]^{1/a}, \quad a, b > 0$
Log-Logistic	(0, ∞)	$\alpha[P/(1-P)]^{\frac{1}{\beta}}, \quad \alpha \,\&\, \beta > 0$
Logistic	(-∞,∞)	$\alpha + \beta \log\left[\frac{P}{(1-P)}\right], \quad \alpha \,\&\, \beta > 0$
Cauchy	(-∞,∞)	$atan[\pi(P-1/2)], \quad a > 0$

T- EE {Uniform} class of probability models: Let Y follow uniform RV with $f_Y(x) = 1$ with QF in Table 1, $Q_Y(p) = P$, the cdf of T-EE {uniform} class corresponding to eq (6) can be obtained as follows

$$F_X(x) = F_T\big(F_R(x)\big) = F_T\Big(\big(1-e^{-\lambda x}\big)^{\alpha}\Big) \tag{9}$$

and the pdf of T-EE {uniform} class corresponding to eq (7) can be represented as

$$f_X(x) = \alpha\lambda e^{-\lambda x}\big(1-e^{-\lambda x}\big)^{\alpha-1} f_T\Big(\big(1-e^{-\lambda x}\big)^{\alpha}\Big) \tag{10}$$

OR

$$f_X(x) = f_R(x) f_T\big(F_R(x)\big)$$

Using eq (8), the corresponding HRF of T-EE {uniform}class becomes

$$h_R(x) = \frac{\alpha\lambda e^{-\lambda x}\big(1-exp(-\lambda x)\big)^{\alpha-1} f_T\Big(\big(1-exp(-\lambda x)\big)^{\alpha}\Big)}{1-F_T\Big(\big(1-exp(-\lambda x)\big)^{\alpha}\Big)} \tag{11}$$

T- EE {Exponential} class of probability models: Let Y follows exponential RV with $f_Y(x) = \beta^{-1}e^{-x/\beta}, x, \beta > 0$ with quantile function in Table 1, $Q_Y(p) = -\beta log(1-P)$, the cdf of T-EE{exponential} class corresponding to eq (6) is obtained as follows

$$F_X(x) = F_T\left(-\beta \log\Big(1-\big(1-e^{-\lambda x}\big)^{\alpha}\Big)\right) = F_T\{-\beta H_R(x)\} \tag{12}$$

The pdf of the T-EE {exponential} class corresponding to eq (7) can be expressed as

$$f_X(x) = \alpha\beta\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1}\frac{f_T\left(-\beta\log\left(1-\left(1-e^{-\lambda x}\right)^{\alpha}\right)\right)}{\left(1-\left(1-e^{-\lambda x}\right)^{\alpha}\right)} \tag{13}$$

$$f_X(x) = \beta h_R(x) f_T\{-\beta H_R(x)\} \tag{14}$$

Where $\left(1-F_R(x)\right)$, $f_R(x)/\left(1-F_R(x)\right), -log\left(1-F_R(x)\right)$ are the survival, HRF and cumulative hazard rate (CHRF) functions, respectively. The corresponding HRF of the T-EE {exponential} class becomes

$$h_X(x) = \frac{\alpha\beta\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1} f_T\left(-\beta\log\left(1-\left(1-e^{-\lambda x}\right)^{\alpha}\right)\right)/\left(\bar{F}_R(x)\right)}{1-F_T\left(-\beta\log\left(1-\left(1-e^{-\lambda x}\right)^{\alpha}\right)\right)} \tag{15}$$

$$h_X(x) = \frac{\beta h_R(x) f_T\{-\beta H_R(x)\}}{1-F_T(-\beta\log\bar{F}_R(x))} \tag{16}$$

From eq. (14) and eq. (16), it can be seen that the pdf and HRF of the T-EE {exponential} class of distributions can be represented in terms of HRF and CHRF of EED.

***T-EE {Weibull} class of probability models*:** Let Y follows a Weibull RV with $f_Y(x) = a/b\,(x/b)^{a-1}exp(-(x/b)^{a})$ and $F_Y(x) = 1-exp(-(x/b)^{a})$, $x \geq 0, a, b > 0$ with the QF in Table 1, $Q_Y(p) = b[-\log(1-P)]^{1/a}$.

The cdf of T-EE {Weibull} class corresponding to eq (6) can be expressed as

$$F_X(x) = F_T\left\{b\left(-\log\left(1-\left(1-e^{-\lambda x}\right)^{\alpha}\right)\right)^{1/a}\right\} \tag{17}$$

Thus, the pdf of the T-EE{Weibull}class is given as

$$f_X(x) = \frac{\alpha\lambda e^{-\lambda x}\left(1-exp(-\lambda x)\right)^{\alpha-1} f_T\left\{b\left(-\log\left(1-\left(1-exp(-\lambda x)\right)^{\alpha}\right)\right)^{1/a}\right\}}{(a/b)\left\{1-\left(1-exp(-\lambda x)\right)^{\alpha}\right\}\left\{-\log\left\{1-\left(1-exp(-\lambda x)\right)^{\alpha}\right\}\right\}^{1-1/a}} \tag{18}$$

$$f_X(x) = b/a\;h_R(x)\{H_R(x)\}^{1/\alpha-1} f_T\left\{b\left(H_R(x)\right)^{1/a}\right\} \tag{19}$$

For $a = 1$, the T-EE {Weibull} class of models approaches to T-EE {exponential} class of models. The corresponding HRF of T-EE (Weibull}class of distribution becomes

$$h_X(x) = \frac{b/a\,h_R(x)\left(H_R(x)\right)^{1/\alpha-1} f_T\left\{b\left(H_R(x)\right)^{1/a}\right\}}{1-F_T\left\{b\left(-\log\left(1-\left(1-exp(-\lambda x)\right)^{\alpha}\right)\right)^{1/a}\right\}} \tag{20}$$

***T-EE {Log-Logistic} class of probability models*:** Let RV Y follows Log-Logistic distribution with pdf $f_Y(x) = (ba^{-b})(x/a)^{b-1}\{1 + (x/a)^b\}^{-2}, x \geq 0, a, b > 0$ with quantile function in Table 1, $Q_Y(p) = a(P/(1-P))^{1/b}$

The cdf of T-EE {Log-Logistics} class corresponding to eq (6)

$$F_X(x) = F_T\left\{a\left(\frac{F_R(x)}{\bar{F}_R(x)}\right)^{1/b}\right\} = F_T\left\{a\left(\left(1 - exp(-\lambda x)\right)^{-\alpha} - 1\right)^{-1/b}\right\} \quad (21)$$

The pdf of the T-EE {Log-Logistic} class of distribution is given as

$$f_X(x) = \frac{a\alpha\lambda e^{-\lambda x}\left(1-exp(-\lambda x)\right)^{-1+\alpha/b} f_T\left\{a\left(\left(1-exp(-\lambda x)\right)^{-\alpha}-1\right)^{-1/b}\right\}}{b\left(\bar{F}_R(x)\right)^{1+1/b}} \quad (22)$$

For $a = 1$ *and* $b = 1$, the pdf of a class of T-EE {Log-Logistic} distribution arising from the odd of exponentiated exponential distribution

$$f_X(x) = \frac{f_R(x)}{\{\bar{F}_R(x)\}^2} f_T\left(\frac{F_R(x)}{\bar{F}_R(x)}\right) \quad (23)$$

Where the corresponding HRF of the T-EE {Log-Logistic} class of distribution is given as

$$h_X(x) = \frac{\alpha\lambda e^{-\lambda x}\left(1-exp(-\lambda x)\right)^{\alpha-1} f_T\left\{\left(\left(1-exp(-\lambda x)\right)^{-\alpha}-1\right)^{-1}\right\}}{\left\{1-\left(1-exp(-\lambda x)\right)^{\alpha}\right\}^2\left\{1-F_T\left\{\left(\left(1-exp(-\lambda x)\right)^{-\alpha}-1\right)^{-1}\right\}\right\}} \quad (24)$$

$$h_X(x) = \frac{f_R(x) f_T(F_R(x)/\bar{F}_R(x))}{\{\bar{F}_R(x)\}^2\{1-F_T\left((F_R(x)/\bar{F}_R(x))\right)\}} \quad (25)$$

***T-EE {Logistic} class of probability models*:** Let the RV "Y" follow Logistic probability distribution with parameters a, b as $f_Y(x) = b^{-1}exp\left((x-a)/b\right)\left[1 + exp\left((x-a)/b\right)\right]^{-2}$having QF in Table 1,

The cdf of T-EE {Logistic} class corresponding to eq (6) can be obtained as follows.

$$F_X(x) = F_T\left\{a + b\log\left(\frac{1}{\left(1-exp(-\lambda x)\right)^{-\alpha}-1}\right)\right\} \quad (26)$$

The pdf of the T-EE {Logistic}class corresponding to eq (7) is given as

$$f_X(x) = b\alpha\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1}\frac{f_T\left(a+b\log\left(\left(1-e^{-\lambda x}\right)^{-\alpha}-1\right)^{-1}\right)}{\left(1-\left(1-e^{-\lambda x}\right)^{\alpha}\right)\left(1-e^{-\lambda x}\right)^{\alpha}} \tag{27}$$

$$f_X(x) = \frac{bh_R(x)}{\bar{F}_R(x)}f_T(a+b\log(F_R(x)/\bar{F}_R(x))) \tag{28}$$

The pdf of the T-EE {Logistic}class of distribution can be used as the weighted hazard function and survival function of EED. Taking $a = 0$ and $b = 1$, through the logit function of the EED, this class of distribution occurs as

$$f_X(x) = \frac{f_R(x)}{F_X(x)(\bar{F}_R(x))}f_T[\log(F_R(x)/\bar{F}_R(x))] \tag{29}$$

$$f_X(x) = \frac{bh_R(x)}{\bar{F}_R(x)}f_T\left(\frac{F_R(x)}{\bar{F}_R(x)}\right) \tag{30}$$

Where the corresponding hazard function of the T-EE {Logistic} class becomes

$$h_X(x) = \frac{b\alpha\lambda e^{-\lambda x}(1-exp(-\lambda x))^{-1}f_T\left(a+b\log\left((1-exp(-\lambda x))^{-\alpha}-1\right)^{-1}\right)}{\left\{1-F_T\left(a+blog\left((1-exp(-\lambda x))^{-\alpha}-1\right)^{-1}\right)\right\}\left(1-(1-exp(-\lambda x))^{\alpha}\right)} \tag{31}$$

$$h_X(x) = \frac{bf_R(x)f_T(a+b\log(F_R(x)/\bar{F}_R(x)))}{\{1-F_T(a+b\log(F_R(x)/\bar{F}_R(x)))\}F_R(x)(\bar{F}_R(x))} \tag{32}$$

***T- EE {Cauchy} class of probability models*:** Let the RV "Y" follow Cauchy distribution with pdf $f_Y(x) = \{a\pi(1+(x/a)^2)\}^{-1}$ QF in Table 1, $Q_Y(p) = atan[\pi(P-0.5)]$

The cdf of T-EE {Cauchy} class, corresponding to eq (6), can be obtained as follows:

$$F_X(x) = F_T[atan\{\pi(F_R(x)-0.5)\}] \tag{33}$$

And the corresponding pdf from eq (7)

$$f_X(x) = a\pi\alpha\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1}sec^2\left(\pi(F_R(x)-0.5)\right)f_T\left(atan\left(\pi(F_R(x)-0.5)\right)\right) \tag{34}$$

Where the corresponding hazard rate function of T-EE {Cauchy} class becomes

$$h_X(x) = \frac{a\pi\alpha\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1}sec^2\left(\pi(F_R(x)-0.5)\right)f_T\left(atan\left(\pi(F_R(x)-0.5)\right)\right)}{1-F_T[atan\{\pi(F_R(x)-0.5)\}]} \tag{35}$$

3. Some statistical measures of the "T-EE{Y}" class of probability models

In this section, several statistical properties of the "T-EE{Y} class", containing quantiles, mode, incomplete moments, entropy, and mean deviation, are presented.

Lemma 1. Let the RV "T" has the pdf $f_T(x)$ then, under the transformation $T = Q_R\left(F_Y(T)\right)$, then X follows the T-EE{Y} class of distribution and hence $X = F^{-1}P(T)$ *or* $X = Q_R\left(F_Y(T)\right)$ *, where* $Q_R(.)$ is the quantile function of the exponentiated exponential distribution and P is the cdf of Y, having a quantile function as Q_Y. Then the X for T-EE{Y} can be obtained as

$$X = \left\{\log\left(1 - \left(F_Y(T)\right)^{1/\alpha}\right)\right\}^{-1/\lambda}$$

or

$$X = -\frac{1}{\lambda}\left\{\log\left(1 - \left(F_Y(T)\right)^{1/\alpha}\right)\right\}$$

Corollary 1. Let the RV "T" with pdf $f_T(x)$, then Lemma 1 can be summarised as

a) If $T \in (0,1)$, then the random variable $X = \left\{\log\left(1 - T^{1/\alpha}\right)\right\}^{-1/\lambda}$ follows the T-EE {uniform} family.

b) If $T \in (0, \infty)$, then the random variable
   i. $X = \left\{\log\left(1 - \left(1 - e^{-(T/\beta)}\right)^{1/\alpha}\right)\right\}^{-1/\lambda}$ follows the T-EE {exponential} family.
   ii. $X = \left\{\log\left\{1 - \left\{1 - e^{-(T/b)^{a}}\right\}^{1/\alpha}\right\}\right\}^{-1/\lambda}$ follows the T-EE {Weibull} family.
   iii. $X = \left\{\log\left(1 - \left(1 + (T/a)^{-b}\right)^{-1/\alpha}\right)\right\}^{-1/\lambda}$ follows the T-EE {log-logistic} family.

c) If $T \in (-\infty, \infty)$, then the random variable
   i. $X = \left\{\log\left(1 - \left(1 + e^{-((T-a)/b)}\right)^{1/\alpha}\right)\right\}^{-1/\lambda}$ follows the T-EE {logistic} family.
   ii. $X = \left\{\log\left(1 - \left(\frac{1}{\pi}\left(tan^{-1}\left(\frac{T}{a}\right) + 0.5\right)\right)^{1/\alpha}\right)\right\}^{-1/\lambda}$ follows the T-EE {Cauchy} family.

Using the fact $T = Q_R\left(F_Y(T)\right)$ gives the relationship between the random variables T and X. By using T, we can generate RV "X", i.e. we can simulate random variable X from random variable T. For example, firstly, the random variable T is simulated from the pdf and then calculating $X = -\frac{1}{\lambda}\left\{\log\left(1 - \left(F_Y(T)\right)^{1/\alpha}\right)\right\}$, the random variable X can be generated which follows the T-EE {Log-Logistic} class of distributions.

Lemma 2. The quantile function of RV "X" is given by $Q_X(\lambda) = Q_R\{F_Y(Q_T(\lambda))\}, 0 < \lambda > 1$, *where* Q_R *and* Q_T are the quantile functions of the RV "R" and "T". Then the quantile function of T-EE{Y}class can be obtained as

$$Q_X(\lambda) = -\frac{1}{\lambda}\log\left(1 - \left(Q_T\left(F_Y(T)\right)\right)^{1/\alpha}\right)$$

Corollary 2. Based on Lemma 1, the QF's for the proposed six sub-models: "(i) T-EE {uniform} (ii) T-EE {exponential} (iii) T-EE {Weibull} (iv) T-EE {Log-Logistic} (v) T-EE {Logistic} (vi) T-EE{Cauchy}" are given as

i) $$Q_X(p) = \left\{\log\left(1 - Q_T(p)^{1/\alpha}\right)\right\}^{-1/\lambda}$$

ii) $$Q_X(p) = \left\{\log\left(1 - \left(1 - e^{-(Q_T(p)/\beta)}\right)^{1/\alpha}\right)\right\}^{-1/\lambda}$$

iii) $$Q_X(p) = \left\{\log\left\{1 - \left\{1 - e^{-(Q_T(p)/b)^{a}}\right\}^{1/\alpha}\right\}\right\}^{-1/\lambda}$$

iv) $$Q_X(p) = \left\{\log\left(1 - \left(1 + (Q_T(p)/a)^{-b}\right)^{-1/\alpha}\right)\right\}^{-1/\lambda}$$

v) $$Q_X(p) = \left\{\log\left(1 - \left(1 - e^{-((Q_T(p)-a)/b)}\right)^{-1/\alpha}\right)\right\}^{-1/\lambda}$$

vi) $$Q_X(p) = \left\{\log 1 - \left(\frac{1}{\pi}\left(tan^{-1}\left(\frac{Q_T(p)}{a}\right) + 0.5\right)\right)^{1/\alpha}\right\}^{-1/\lambda}$$

Theorem 1. The mode of "T-EE{Y} class" is obtained the solving of the equation

$$\left(1 - e^{-\lambda x}\right)^{-\alpha-2}\left(e^{\lambda x} - \alpha\right) = \alpha\left[\frac{\acute{\acute{Q}}_Y\left(F_R(x)\right)}{\acute{Q}_Y\left(F_R(x)\right)} + \frac{\acute{f}_T\left(Q_Y\left(F_R(x)\right)\right)}{\acute{f}_T\left(Q_Y\left(F_R(x)\right)\right)}\acute{Q}_Y\left(F_R(x)\right)\right]$$

The above equation can be solved to find the mode of the proposed class of probability models.

Theorem 2. Let RV "X" has the T-EE{Y} class of distribution with *pdf* $f_x(x)$, then nth moment of *X is* $E(X^n)$, then

$$E(X^n) \leq E(R^n)E\left[\left(1 - F_Y(T)\right)^{-1}\right]$$

Where $E(X^n)$ are the nth moments of the pdf in eq (6) and $1 - \mathrm{F_Y(T)}$ is the survival function with cdf (F_Y)with random variable T and random variable T having the pdf as f_T. If moments exist, then the nth moments for T-EE{Y}class are defined as:

$$E(X^n) = \frac{(-1)^n \alpha}{\lambda^n} \frac{\partial^n}{\partial p^n} B(\alpha, p+1-\alpha)|_{p=\alpha} E\left[\left(1 - F_Y(T)\right)^{-1}\right]$$

Corollary 3. From Theorem 2 (the upper bound of moments), the nth moments for the following classes: " i). T-EE{uniform} ii). T-EE{exponential } iii). T-EE{weibull}iv). T-EE{log − logistic} v). T-EE{logistic} vi). T-EE{cauchy}" are given as

i) If $X \sim \mathrm{T-EE\{uniform\}}$, then $E(X^n) = \frac{(-1)^n \alpha}{\lambda^n} \frac{\partial^n}{\partial p^n} B(\alpha, p+1-\alpha)|_{p=\alpha} E\left(\frac{b-a}{b-T}\right)$.

ii) If $X \sim \mathrm{T-EE\{exponential\}}$, then $E(X^n) = \frac{(-1)^n \alpha}{\lambda^n} \frac{\partial^n}{\partial p^n} B(\alpha, p+1-\alpha)|_{p=\alpha}$ $M_T(1/\beta)$.

iii) If $X \sim T - EE\{\mathrm{Weibull}\}$, then $E(X^n) = \frac{(-1)^n \alpha}{\lambda^n} \frac{\partial^n}{\partial p^n} B(\alpha, p+1-\alpha)|_{p=\alpha}$ $M_{T^a}(1/b^a)$.

iv) If $X \sim \mathrm{T-EE\{Log-Logistic\}}$, then $E(X^n) = \frac{(-1)^n \alpha}{\lambda^n} \frac{\partial^n}{\partial p^n} B(\alpha, p+1-\alpha)|_{p=\alpha} 1 + a^{-b} E\left(T^b\right)$.

v) If $X \sim \mathrm{T-EE\{Logistic\}}$, then $E(X^n) = \frac{(-1)^n \alpha}{\lambda^n} \frac{\partial^n}{\partial p^n} B(\alpha, p+1-\alpha)|_{p=\alpha} \{1 + e^{-a/b} M_t(1/b)\}$.

vi) If $X \sim \mathrm{T-EE\{Cauchy\}}$, then $E(X^n) = \frac{(-1)^n \alpha}{\lambda^n} \frac{\partial^n}{\partial p^n} B(\alpha, p+1-\alpha)|_{p=\alpha}$ $\left\{\frac{2\pi}{\pi - 2E(\tan^{-1}(T/a))}\right\}$.

Theorem 3. The Shannon entropy for the class of T-EE{Y} distributions is given by

$$\eta_x = \eta_T + E(\log f_Y(T)) - \log(\alpha\lambda) - (\alpha - 1)E\left(\log\left(1 - e^{-\lambda x}\right)\right) + \lambda\mu_x,$$

Where η_x , η_T are the Shannon entropies for X and T, respectively*, and* $\mu_x = E(X)$

Proof. For an RV "X", Shannon's entropy is

$$\eta_x = \mathbb{E}\{-\log f_X(x)\}$$

From Lemma 1, using the relationship $T = Q_y[F_R(x)]$

$$f_X(x) = f_R(x) \frac{f_T(t)}{f_Y(t)}$$

Taking $(-log)$ and applying expectations

$$E(-\log f_X(x)) = E(-\log f_T(T)) + E(\log f_Y(T)) - E(\log f_R(X))$$

$$\eta_x = \eta_T + E(\log f_Y(T)) - \log(\alpha\lambda) - (\alpha - 1)E\left(\log\left(1 - e^{-\lambda x}\right)\right) + \lambda\mu_x$$

Corollary 4. From Theorem 3, Shannon entropy for the following classes*: " i). T-EE*$\{uniform\}$ *ii). T-EE*$\{exponential\}$ *iii). T-EE*$\{weibull\}$ *iv). T-EE*$\{log - logistic\}$ *v). T-EE*$\{logistic\}$ *vi). T-EE*$\{cauchy\}$*" are obtained as*

i) $\eta_x = \eta_T - \log(\alpha\lambda) - (\alpha - 1)E\left(\log\left(1 - e^{-\lambda x}\right)\right) + \lambda\mu_x$

ii) $\eta_x = \eta_T - E(\log\beta) - \frac{1}{\beta}E(T) - \log\alpha\lambda - (\alpha - 1)E\left(\log\left(1 - e^{-\lambda x}\right)\right) + \lambda\mu_x$

iii) $\eta_x = \eta_T + \log(a/b) + (a - 1)E(\log(T/b)) - 1/b^{-a}\, E(T^a) - \log(\alpha\lambda) - (\alpha - 1)E\left(\log\left(1 - e^{-\lambda x}\right)\right) + \lambda\mu_x$

iv) $\eta_x = \eta_T + \log\left(ba^{-b}\right) + (b - 1)E(\log(T/a)) - 2E\left(\log\left(1 + (T/a)^b\right)\right) - \frac{1}{\beta}E(T) - \log(\alpha\lambda) - (\alpha - 1)E\left(\log\left(1 - e^{-\lambda x}\right)\right) + \lambda\mu_x$

v) $\eta_x = \eta_T + ab^{-1} - \log b + b^{-1}E(\log(T)) - 2E\left(\log\left(1 + exp\left(-\left(\frac{T-a}{b}\right)\right)\right)\right) - \log(\alpha\lambda) - (\alpha - 1)E\left(\log\left(1 - exp(-\lambda x)\right)\right) + \lambda\mu_x$

vi) $\eta_x = \eta_T - \log(a\pi) - E(\log(1 + (T/a)^2)) - \log(\alpha\lambda) - (\alpha - 1)E\left(\log\left(1 - e^{-\lambda x}\right)\right) + \lambda\mu_x$

Theorem 4. Let the mean deviation from the mean and median be D_μ and D_M respectively, which measure the dispersion of the population from the Mean and Median, respectively. Then D_μ and D_M for the class of "T-EE{Y} class", respectively, are obtained by

$$D_\mu = 2\mu F_T\left(Q_Y\left(F_R(\mu)\right)\right) - 2I_\mu$$

$$D_M = \mu - 2I_M$$

Where $I_p = \int_0^p x f_x(x)dx$ is the incomplete moment of the distribution. From Lemma 1, we have $X = Q_R\left(F_Y(T)\right)$ so $I_p = \int_0^{Q_Y\left(F_R(P)\right)} Q_R\left(F_Y(\mu)\right) f_T(\mu)d\mu$

Summarisation of the Theorem 4 is as follows:

i Y~Uniform,

$$D_\mu = 2\mu F_T\left(Q_Y\left(F_R(\mu)\right)\right) - 2\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j\lambda}S_u(\mu)$$

$$D_M = \mu - 2\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j\lambda}S_u(M)$$

Where $S_Y(p) = \int_0^{(1-e^{-\lambda p})^{\alpha}}\left\{-(w)^{1/\alpha}\right\}^j f_T(w)dw.$

ii $Y\sim$exponential,

$$D_\mu = 2\mu F_T\left(Q_Y\left(F_R(\mu)\right)\right) - 2\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j\lambda}S_e(\mu),$$

$$D_M = \mu - 2\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j\lambda}S_e(M)$$

Where $S_Y(p) = \int_0^{-\beta log\left(1-(1-e^{-\lambda p})^{\alpha}\right)}\left\{-\left(1-e^{-w/\beta}\right)^{1/\alpha}\right\}^j f_T(w)dw.$

iii $Y\sim$Weibull,

$$D_\mu = 2\mu F_T\left(Q_Y\left(F_R(\mu)\right)\right) - 2\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j\lambda}S_e(\mu),$$

$$D_M = \mu - 2\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j\lambda}S_e(M),$$

Where $S_Y(p) = \int_0^{-b\left\{log\left(1-(1-e^{-\lambda x})^{\alpha}\right)\right\}^{1/a}}\left\{-\left(1-e^{-w/\beta}\right)^{1/\alpha}\right\}^j f_T(w)dw.$

iv $Y\sim$Log-Logitic,

$$D_\mu = 2\mu F_T\left(Q_Y\left(F_R(\mu)\right)\right) - 2\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j\lambda}S_e(\mu),$$

$$D_M = \mu - 2\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j\lambda}S_e(M),$$

Where $S_Y(p) = \int_0^{a\left(\left(1-e^{-\lambda x}\right)^{-\alpha}-1\right)^{-1/b}}\left\{-\left(1+(w/a)^{-b}\right)^{-1/\alpha}\right\}^j f_T(w)dw.$

v $Y\sim$Logistic,

$$D_\mu = 2\mu F_T\left(Q_Y\left(F_R(\mu)\right)\right) - 2\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j\lambda}S_e(\mu),$$

$$D_M = \mu - 2\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j\lambda}S_e(M),$$

Where $S_Y(p) = \int_0^{a-blog\left(\left(1-e^{-\lambda x}\right)^{-\alpha}-1\right)}\left\{-\left(1-e^{-((w-a)/b)}\right)^{-1/\alpha}\right\}^j f_T(w)dw.$

vi $Y\sim$Cauchy,

$$D_\mu = 2\mu F_T\left(Q_Y\left(F_R(\mu)\right)\right) - 2\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j\lambda}S_e(\mu),$$

$$D_M = \mu - 2\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j\lambda}S_e(M),$$

where $S_Y(p) = \int_0^{aatan\left(\pi\left(\left(1-e^{-\lambda x}\right)^{\alpha}-0.5\right)\right)} \left\{-\left(\frac{1}{\pi}\left(tan^{-1}(w/a)+0.5\right)\right)^{1/\alpha}\right\}^{j} f_T(w)dw.$

4. A Few Members of the T-EE{Y} Class of Probability Models

This section inspects four members of the "T-EE{Y} class" of probability models. To generate these four members, we have chosen distributions of T and Y. The first one is a member of T-EE{W} class of probability models, the second and third one are the members of T-EE {LL} class of distributions, and the fourth one is the subset of T-EE{L} class of distributions.

4.1. The Weibull-EE {Weibull} distribution

Let T be a Weibull random variable with the parameters $\theta and\ \gamma$. The pdf and cdf are as follows $f_T(x) = \theta\gamma t^{\gamma-1}e^{-\theta t^{\gamma}}$ and $F_T(x) = 1 - e^{-\theta t^{\gamma}}$, where $x \geq 0, \theta, \gamma > 0$, with the quantile function in the table, $Q_T(p) = \left\{-\frac{1}{\theta}\log(1-P)\right\}^{1/\gamma}$

Using eq (17) the cdf of the RV "X" with W-EE{W} becomes

$$F_X(x) = 1 - \exp\left\{-\theta b^{\gamma}\left\{-\log\left(1-\left(1-e^{-\lambda x}\right)^{\alpha}\right)\right\}^{\gamma/a}\right\} \qquad (36)$$

Using eq (18) with $\gamma/a = c$, the corresponding pdf of W-EE{W} can be obtained as

$$f_X(x) = \frac{\alpha\lambda\theta c b^{\gamma} e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1}\left\{-\log\left\{1-\left(1-e^{-\lambda x}\right)^{\alpha}\right\}\right\}^{c-1}\ e^{-\theta b^{\gamma}\left\{-\log\left(1-\left(1-e^{-\lambda x}\right)^{\alpha}\right)\right\}^{c}}}{\{1-(1-e^{-\lambda x})^{\alpha}\}}$$

To minimise the effect of irrelevant shape and scale parameters take $\theta = 1, b = 1$

$$f_X(x) = \frac{\alpha\lambda c e^{-\lambda x}\left\{-\log\left(1-\left(1-exp(-\lambda x)\right)^{\alpha}\right)\right\}^{c-1} e^{-\left\{-\log\left(1-\left(1-exp(-\lambda x)\right)^{\alpha}\right)\right\}^{c}}}{\left(1-exp(-\lambda x)\right)} \qquad (37)$$

For $c = 1$, the W-EE{W}distribution reduces to the exponentiated exponential distribution and for $\alpha = 1$, the W-EE{W}distribution reduces to the exponential distribution.

The quantile function: Lemma 2 gives the QF of the W-EE{W} distribution. So, QF of W-EE{W}distribution is given as

$$Q_X(p) = -\frac{1}{\lambda}\log\left\{1-(1-(1-P)^{c})^{1/\alpha}\right\} \qquad (38)$$

HRF of W-EE{W}distribution is as follows

$$h_X(x) = \frac{\alpha\lambda c e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{-1}\left\{\log\left(1-\left(1-e^{-\lambda x}\right)^{\alpha}\right)\right\}^{-c+1} e^{-\left\{-\log\left(1-\left(1-e^{-\lambda x}\right)^{\alpha}\right)\right\}^{c}}}{\exp\left\{\log\left(1-\left(1-e^{-\lambda x}\right)^{\alpha}\right)\right\}^{-c}} \tag{39}$$

Figure 1 provides graphs of the three-parameter W-EE{W} distribution for different values of α, λ, c. The density plots show that by increasing the value of α the curve of the density changes from skewed to symmetrical. By decreasing the value of λ, the density changes from rightly skewed to symmetrical as well and it also stretches the density by decreasing the value of λ. Also indicate that the parameter c, which is a new parameter, changes the shape of the distribution as well as the kurtosis. From all these figures, we conclude that the W-EE{W}distribution is capable of being right skewed, left skewed and unimodal,with only three parameters. So the distribution is very flexible and versatile also applicable for different types of data sets in real life.

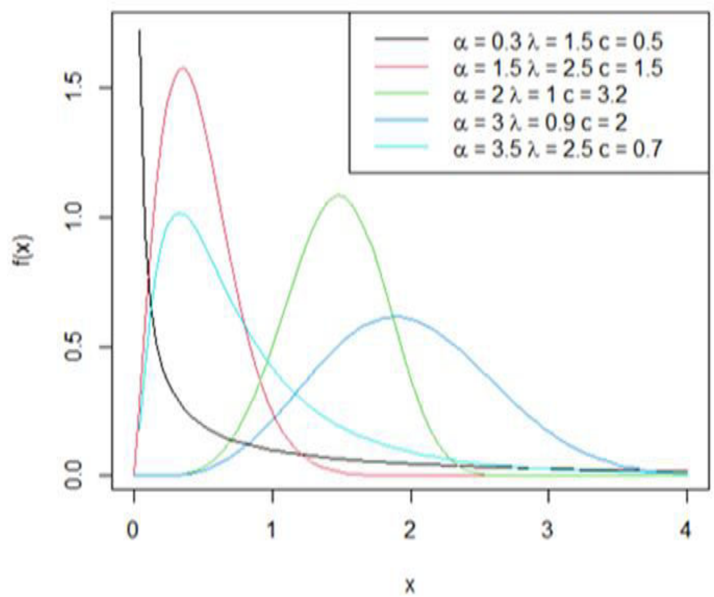

Figure 1: density plots of W-EE{W} distribution for different values of α, λ *and* c

4.2. Weibull-EE {Log-Logistic} distribution

Let T be a Weibull random variable with the parameters $\theta and\ \gamma$. The pdf and cdf are as follows $f_T(x) = \theta\gamma t^{\gamma-1} e^{-\theta t^{\gamma}}$ and $F_T(x) = 1 - e^{-\theta t^{\gamma}}$, where $x \geq 0; \theta, \gamma > 0$, with the quantile function in table, $Q_T(p) = \left\{-\frac{1}{\theta}\log(1-P)\right\}^{1/\gamma}$

Using eq (21) with $\gamma/b = c$, the cdf of the RV "X" follows W-EE {LL} distribution, is given as

$$F_X(x) = 1 - \exp\left\{-\left(\left(1 - e^{-\lambda x}\right)^{-\alpha} - 1\right)^{-c}\right\} \qquad (40)$$

Also, using eq (22), the pdf of W-EE{LL} is obtained as

$$f_X(x) = \frac{\alpha\lambda\theta c a^{\gamma} e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha c-1} e^{-\theta a^{\gamma}\left(\left(1-e^{-\lambda x}\right)^{-\alpha}-1\right)^{-c}}}{\left(1-\left(1-e^{-\lambda x}\right)^{\alpha}\right)^{c+1}} \qquad (41)$$

To minimise the effect of irrelevant shape and scale parameters, put $\theta = 1, a = 1$

$$f_X(x) = \frac{\alpha\lambda c e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha c-1}}{\left(1-\left(1-e^{-\lambda x}\right)^{\alpha}\right)^{c+1}} exp\left\{-\left(\left(1 - e^{-\lambda x}\right)^{-\alpha} - 1\right)^{-c}\right\} \qquad (42)$$

When $\gamma = 1$, and $c = 1/b$ eq (41) becomes E-EE {LL} distribution. When $\alpha = 1$, eq (41) reduces W-E {LL} distribution.

The QF of the W-EE{LL} distribution is computed as

$$Q_X(p) = -\frac{1}{\lambda}\log\left\{1 - \left(1 + \left(\{-\log(1-P)\}^{-1/c}\right)\right)^{-1/\alpha}\right\} \qquad (43)$$

HRF of W-EE{LL} is as follows

$$h_X(x) = \frac{\alpha\lambda c e^{-\lambda x}\left(1-exp(-\lambda x)\right)^{\alpha c-1}}{\left(1-\left(1-exp(-\lambda x)\right)^{\alpha}\right)^{c+1}\left[\exp\left\{-\left(\left(1-exp(-\lambda x)\right)^{-\alpha}-1\right)^{c}\right\}-1\right]} \qquad (44)$$

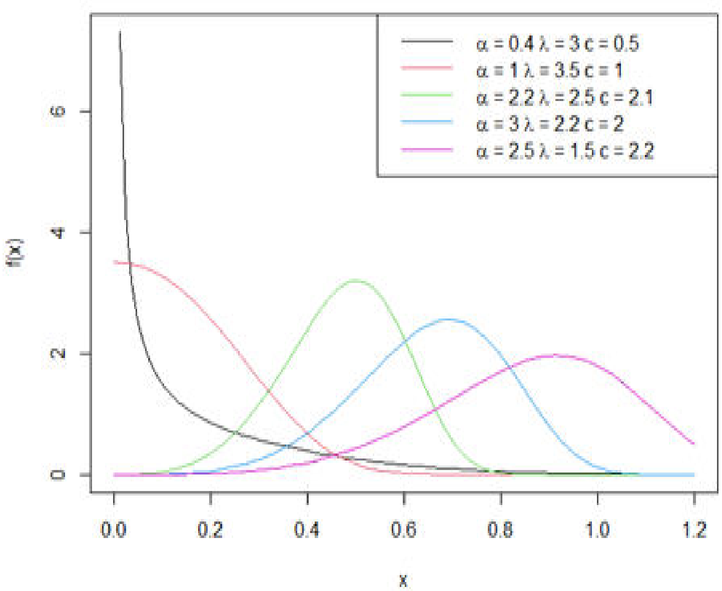

Figure 2: density plots of W-EE{LL}distribution for different values of α, λ, c

In Figure 2, the plots of the density functions of the 3-parameter W-EE{LL}distribution for different values of the parameters (α, λ, c). It shows that with increasing value of λ, the kurtosis changes from leptokurtic to platykurtic. There is the effect of changing the parameter c which is a new parameter. The plots indicate that by increasing the value of c, the distribution changes its shape and after c=1, the density becomes stable, but the kurtosis continues to change. The figure shows the combined effect of all 3 parameters. This proves that W-EE{W}distribution is suitable for several types of data.

4.3. Gamma-EE {Log-logistic} distribution

Let T be a Gamma random variable with the parameters θ and r. The pdf and cdf are as follows $f_t(x) = \frac{r^\theta}{\Gamma(\theta)} t^{\theta-1} e^{-rt}$ and $F_t(x) = \frac{\gamma(\theta,t)}{\Gamma(\theta)}$,where $x \geq 0, \theta, > 0$

Using eq (21), the cdf of the RV "X" which follows the W-EE {LL} distribution is given as

$$F_X(x) = \frac{\gamma\left(\theta,\left(\left(1-e^{-\lambda x}\right)^{-\alpha}-1\right)^{-1/b}\right)}{\Gamma(\theta)} \tag{45}$$

Also, using eq (22), the pdf of W-EE {LL} is obtained. To minimise the irrelevant shape and scale parameter, let, $a = 1$

$$f_X(x) = \frac{\alpha\lambda e^{-\lambda x}\left(\left(1-exp(-\lambda x)\right)\right)^{\alpha\theta/b-1} e^{-\left(\left(1-exp(-\lambda x)\right)^{-\alpha}-1\right)^{-1/b}}}{b\,\Gamma(\theta)\,\left(1-\left(1-exp(-\lambda x)\right)^{\alpha}\right)^{1+\theta/b}} \tag{46}$$

When $\alpha = 1$, eq (46) reduces the Ga-E{LL}distribution.

HRF of Ga-EE{W}distribution is as follows

$$h_X(x) = \frac{\alpha\lambda e^{-\lambda x}\left(\left(1-exp(-\lambda x)\right)^{\alpha}\right)^{\theta/b-1/\alpha} e^{-\left(\left(1-exp(-\lambda x)\right)^{-\alpha}-1\right)^{-1/b}}}{b\left(1-\left(1-exp(-\lambda x)\right)^{\alpha}\right)^{1+\theta/b}\left\{\Gamma(\theta)-\gamma\left(\theta,a\left(\left(1-exp(-\lambda x)\right)^{-\alpha}-1\right)^{-1/b}\right)\right\}} \tag{47}$$

Figure 3 are the plots of the density functions of the 4-parameter Ga-EE{LL}distribution for different values of the parameters $(\alpha, \lambda, b, \theta)$. It shows that by increasing the value of α, the density shifts to the right. It presents that by increasing λ and b the peakness decreases and spread increases but λ needs a very small change, whereas b must change with a large magnitude to change the density. Initially, the ordinate shows a rapid decrease as λ and b increase, the ordinates decrease with a decreasing rate. Plots in the figure are the effect of changing the parameter θ. The plots indicate that by increasing the value of θ, the Figure is the combined effect of all 4 parameters.

Which indicates that Ga-EE{LL}distribution is a very versatile distribution and a good fit for different types of data

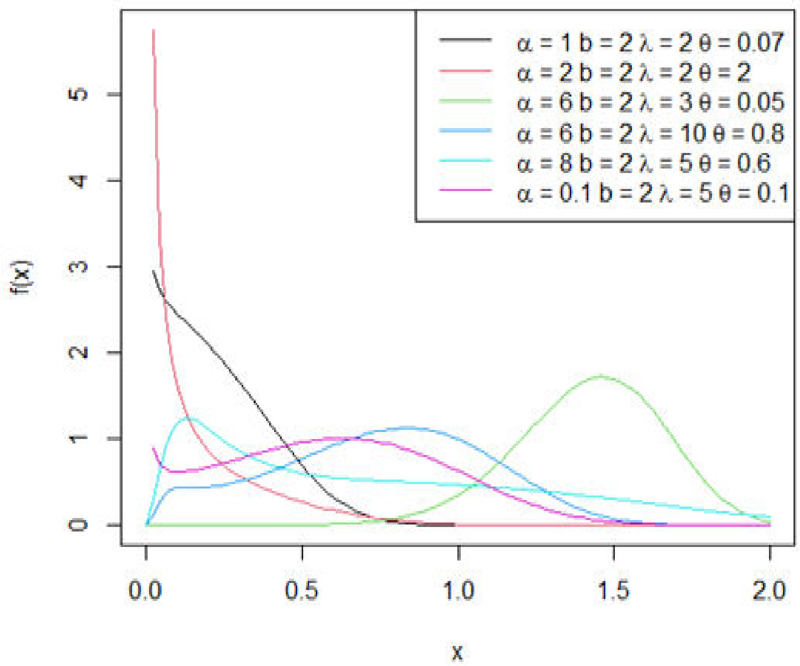

Figure 3: density plots of Ga-EE{LL}distribution for values of α, b, λ *and* θ.

4.4. Normal-EE {Logistic} distribution

Let RV "T" have the normal distribution with the parameters μ and σ. The pdf and cdf of the random variable T are as follows $f_T(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$and $F_T(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$,where $x \geq 0, \mu, \sigma > 0$

Using eq (26), the cdf of the RV "X" which follows the N-EE {L} distribution, is given as

$$F_X(x) = \Phi\left(\frac{\left(-\log\left(\left(1-e^{-\lambda x}\right)^{-\alpha}-1\right)\right)-\mu}{\sigma}\right) \tag{48}$$

Also, using eq (27), the pdf of N-EE{L} can be obtained. To minimise the effect of additional shape and scale parameter, let $a = 0$ *and* $b = 1$,

$$f_X(x) = \frac{\alpha\lambda e^{-\lambda x} e^{-\frac{1}{2\sigma^2}\left(\left(-\log\left(\left(1-e^{-\lambda x}\right)^{-\alpha}-1\right)\right)-\mu\right)^2}}{\sigma\sqrt{2\pi}\left\{1-\left(1-e^{-\lambda x}\right)^{\alpha}\right\}\left(1-e^{-\lambda x}\right)} \tag{49}$$

Figure 4 shows the plots of probability density functions of the N-EE{L}distribution. The figure shows that with the initially small value of α, the density is U-shaped like $\alpha < 1$, when $\alpha > 1$, the shape of the distribution becomes heavily rightly skewed and with increasing value of α. Figures describe that for small values of λ , the distribution is flat, and for large values of λ, the peak of the distribution becomes high. By increasing the value of μ, the density stretches out. The figure shows that the N-EE{L} is also useful for bimodal data sets. This concludes that the distribution is very flexible and useful for different types of data sets, like left skewed, right skewed, unimodal and bimodal as well.

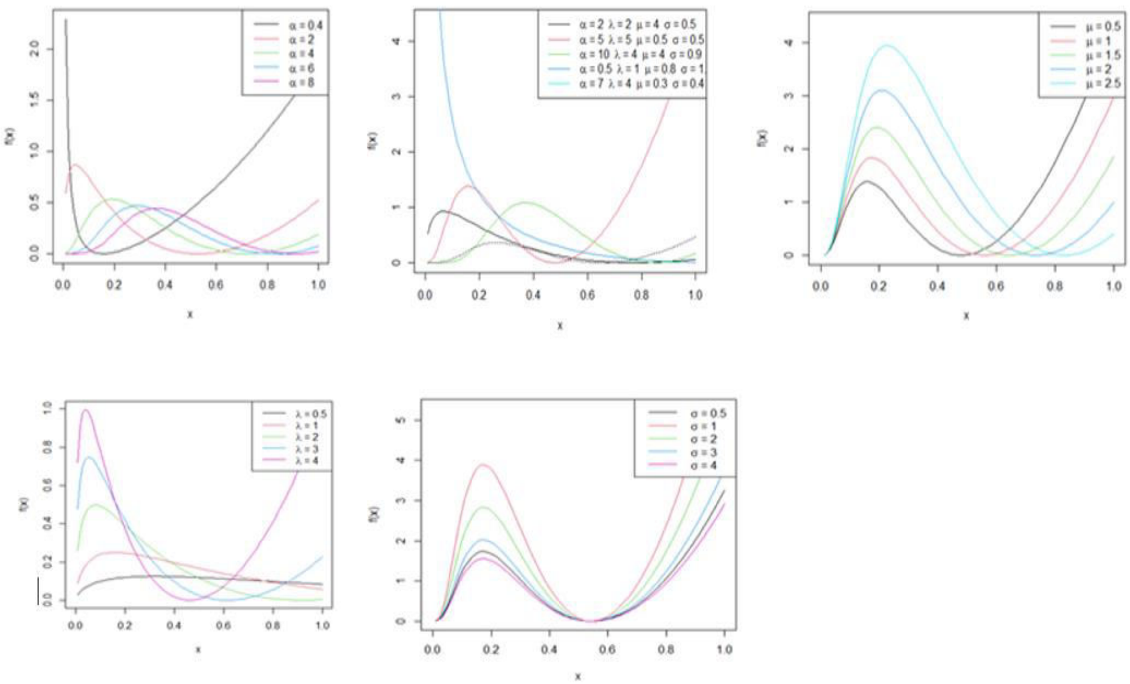

Figure 4: density plots of N-EE{L}distribution for values of α, λ, μ and σ

5. Parameter estimation for the members of T-EE{Y} class of probability models

In this section maximum likelihood estimation method (MLE) has been used to estimate the unknown parameter of the W-EE {W} distribution, W-EE {LL} distribution, Ga-EE {LL} distribution, N-EE {L} distribution. A simulation study of the ML estimators is also carried out to check the efficacy of MLEs.

5.1. The W-EE{W} distribution

Let $X_{1,}X_{2,}\ldots, X_n$ be a r.s from W-EE {LL}distribution. The log likelihood function is given as

$$\log L = nln(\alpha\lambda c) - \lambda\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} ln\left(1 - e^{-\lambda x_i}\right)^{\alpha-1} - \sum_{i=1}^{n}\left(-log\left(\left(1 - \left(1 - e^{-\lambda x}\right)^{\alpha}\right)\right)\right)^{c} + \sum_{i=1}^{n} log\left(-log\left(1 - \left(1 - e^{-\lambda x}\right)^{\alpha}\right)\right)^{c-1} - \sum_{i=1}^{n} log\left(\left(1 - \left(1 - e^{-\lambda x}\right)^{\alpha}\right)\right)$$

(50)

5.2. The W-EE {LL} distribution

Let $X_1, X_2, \ldots, X_n$ be a r.s from W-EE {LL}distribution. The log likelihood function is given as

$$\log L = nln(\alpha\lambda c) - \lambda \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} ln\left(1 - e^{-\lambda x_i}\right)^{\alpha c-1} - \sum_{i=1}^{n}\left(\left(1 - e^{-\lambda x}\right)^{-\alpha} - 1\right)^{-c} - \sum_{i=1}^{n} ln\left(\left(1 - \left(1 - e^{-\lambda x}\right)^{\alpha}\right)^{c+1}\right) \quad (51)$$

5.3. The Ga-EE {LL} distribution

Let $X_1, X_2, \ldots, X_n$ be a r.s of size n from Ga-EE {LL}. The log likelihood function is given as

$$\log L = n\log\left(\frac{(a*\lambda)}{b\Gamma(\theta)}\right) - \lambda \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} ln\left(1 - e^{-\lambda x_i}\right)^{\left(\alpha\theta/b\right)-1} - \sum_{i=1}^{n}\left(\left(1 - e^{-\lambda x}\right)^{-\alpha} - 1\right)^{-\left(1/b\right)} - \sum_{i=1}^{n} ln\left(\left(1 - \left(1 - e^{-\lambda x}\right)^{\alpha}\right)^{\theta/b+1}\right) \quad (52)$$

5.4. N-EE{L}distribution

Let $X_1, X_2, \ldots, X_n$ be a r.s of size n from the N-EE{L}distribution. The log likelihood function is given as

$$\log L = nlog\left(\frac{(a*\lambda)}{\sigma\sqrt{2\pi}}\right) - \lambda \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} log\left(1 - e^{-\lambda x_i}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(-log\left(\left(\left(1 - e^{-\lambda x}\right)^{-\alpha} - 1\right)\right) - \mu\right)^2 - \sum_{i=1}^{n} ln\left(\left(1 - \left(1 - e^{-\lambda x}\right)^{\alpha}\right)\right) \quad (53)$$

The ML estimates for all the parameters involved in the above equations (50), (51), (52), and (53) are those values of the parameters for which the log likelihood is as maximum as possible. The estimated parameters of the W-EE{W}, W-EE{LL}, Ga-EE{LL}, N-EE{L}, distributions are the solution of these equations. Some statistical software is available which can be used to get the estimates by Newton-Raphson iterative method.

5.5. Simulations

To estimate the behaviour and performance of MLEs, a Monte Carlo simulation study has been conducted for the parameter of W-EE{W} and W-EE{LL} distribution. For this purpose, random samples of different sizes (n=25, 50, 100, 200, 500) are generated from W-EE{W} and W-EE{LL} distributions. Four different sets of combinations of parameters, i.e.($\hat{\alpha} = 1,1.5,2, \hat{\lambda} = 0.05,0.5, \hat{b} = 0.5,1,1.5,$) and ($\hat{\alpha} = 0.3,0.5,2.5,3.5, \hat{\lambda} = 0.03,0.04,0.05,,0.5, \hat{c} = 1,1.5,2.5$)and fixing the remaining parameters of W-EE{W} and W-EE{LL} distribution respectively. 500 simulations were performed for each combination of parameters and each sample size.

Tables 2 & 3 present the actual and estimated values of the parameters along with their standard errors for the W-EE{W} and W-EE{LL} respectively. The values in Table 2 indicate that for small sample sizes standard error of estimates is relatively large and it reduces as sample sizes increase. The efficacy of MLEs increases with an increase in sample size, which shows that the ML estimation method is a suitable method for estimation of unknown parameters

Table 2:
MLEs and their standard errors for **W-EE{W}** parameters

Actual values				Estimated values			Standard error		
α	λ	c	n	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{b}$
			25	1.3379	0.5884	1.2028	1.0900	0.4046	0.5495
			50	1.534	0.5346	1.1351	0.72213	0.1945	0.2832
1.5	0.5	1.5	100	1.4476	0.4492	1.0114	0.7129	0.1690	0.2737
			200	1.6350	0.5200	1.0689	0.4269	0.0999	0.1466
			500	1.5667	0.5148	0.9844	0.2880	0.0691	0.0962
			25	2.0478	0.6564	1.101	2.1451	0.4626	0.5914
			50	2.3147	0.6907	1.1730	1.2493	0.2420	0.3095
2	0.5	1	100	1.8115	0.5351	1.2022	0.8108	0.1762	0.2844
			200	1.9432	0.6696	1.1391	0.5352	0.1288	0.1613
			500	1.8300	0.5548	1.0925	0.3262	0.0696	0.1006
			25	1.5100	0.5236	1.0429	1.3924	0.3647	0.5209
			50	1.4011	0.6160	1.0016	0.9853	0.3356	0.3921
1.5	0.5	1	100	1.4222	0.4710	0.9950	0.6615	0.1678	0.2530
			200	1.4248	0.4742	0.9124	0.4814	0.1199	0.1683
			500	1.4314	0.4771	0.9746	0.2678	0.0679	0.0992
			25	1.0871	0.0533	0.8032	1.0122	0.0533	0.4796
			50	0.8147	0.0442	0.5750	0.9011	0.0442	0.3908
1	0.05	0.5	100	0.8745	0.0317	0.7280	0.5361	0.0317	0.2714
			200	1.0728	0.0166	0.7609	0.3845	0.0166	0.1561
			500	0.9882	0.0116	0.7927	0.2429	0.0116	0.1149

6. Application of the selected members of T-EE{Y} class

In this section, the applications of the T-EE{Y} class of probability models have been presented by four of its proposed members. The four proposed members, named W-EE{W}, W-EE{LL}, Ga-EE{W}, N-EE{L}, have been fitted to three real data sets. The maximum likelihood method is used to investigate the parameters of the model. The goodness of fit statistics including AIC (Akaike information criterion), the BIC (Bayesian information criterion), the CAIC (consistent Akaike information criteria) and the HQIC (Hannan Quinn information criterion) has been obtained to show the efficacy of the models and compared to the existing distributions The smaller value of the statistic shows the better fit of the model for the data. The necessary computations for the estimated models are carried out in the R-language.

Table 3:
MLEs and their standard errors for W-EE{LL} parameters

Actual values				Estimated values			Standard error		
α	λ	c	n	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{c}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{c}$
				0.2750	0.0151	2.6853	0.2403	0.0343	2.0985
			50	0.2696	0.0155	3.1784	0.1697	0.0259	1.8056
0.3	0.04	2.5	100	0.2876	0.0173	2.7174	0.1422	0.0214	1.1902
			200	0.2393	0.0122	2.7418	0.0489	0.0073	0.5234
			500	0.2478	0.0129	2.997	0.03484	0.0052	0.3916
			25	2.5151	0.0513	1.4859	8.6781	0.1096	2.6032
			50	2.4642	0.0512	1.5071	8.0407	0.1038	2.4483
2.5	0.05	1.5	100	2.5009	0.0502	1.5200	5.6933	0.0706	1.7293
			200	2.4854	0.0499	1.4818	3.3416	0.0417	1.0082
			500	2.4977	0.0500	1.4931	2.6632	0.0331	0.8083
			25	3.5378	0.5940	0.9009	5.3356	0.4974	0.7028
			50	3.7699	0.5992	0.8724	3.7941	0.3239	0.4403
3.5	0.5	1	100	3.3501	0.5630	1.0485	2.8762	0.2677	0.4401
			200	3.6423	0.5909	1.0749	2.2142	0.1922	0.3174
			500	3.8003	0.5982	0.9518	1.3300	0.1108	0.1624
			25	0.6827	0.0392	1.6227	0.7562	0.05637	1.3298
			50	0.5074	0.0328	1.3406	0.5192	0.0531	1.0799
0.5	0.03	1.5	100	0.6433	0.0477	1.2293	0.3637	0.0366	0.5319
			200	0.5173	0.03406	1.5548	0.3566	0.0341	0.8290
			500	0.5027	0.0299	1.4444	0.2895	0.0299	0.6474

6.1. Bladder cancer data

In this first application, the data on the time of remission (in months) of 128 cancer patients has been used. The bladder cancer data is heavily right-skewed. [16] fitted this data set to their proposed family of generalized Lomax Distribution, based on the "T-R{Y} class" that is proposed by [14] We have compared proposed members of the class of distributions with the following probability models: “Gamma Lomax Log Logistic(G-L{LL}), Exponentiated Weibull Lomax Exponential distribution (EW-L{E}), Normal Lomax Cauchy (N-L{C}) distributions”. The MLEs, standard errors and goodness of fit statistics for the proposed and competitive distributions for bladder cancer data have been shown in Tables 4 and 5. The results in Table 4 & 6 indicate that both the proposed members of the class of distributions W-EE{W}, Ga-EE{LL} distributions perform outstandingly and Figure 5 shows the comparative density plot on bladder cancer data. The data is as follows:

Table 4:
MLEs, standard error and -2logL of bladder cancer data

Distributions	Estimates (standard error)								
	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{c}$	$\hat{a}$	$\hat{b}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\mu}$	$\hat{\sigma}$
W-EE{W}	0.1788 (0.0073)	0.0023 (0.0003)	2.4876 (0.1653)	-	1.000	1.000	-	-	-
W-EE{LL}	0.1847 (0.0071)	0.0023 (0.0002)	2.7277 (0.1771)	-	-	1.000	-	-	-
Ga-EE{LL}	37.2575 (22.0264)	0.1299 (0.0180)	-	10	5.0613 (1.4234)	0.1851 (0.0941)	10	-	-
N-EE{L}	0.2062 (0.1660)	0.0073 (0.0030)	-	-	1.000	-	-	0.1200 (1.0459)	0.4553 (0.1303)

Table 5:
The statistics AIC, BIC, AICC and HQIC for bladder cancer data

Distribution	k	-2logL	AIC	BIC	AICC	HQIC
W-EE{W}	3	820.74	826.74	835.30	826.93	830.22
Ga-EE{LL}	4	819.37	827.37	838.79	827.70	832.01
G-L{LL}	4	821.6	829.6	841.01	829.93	834.24
EW-L{E}	4	819.9	827.9	839.31	828.23	832.54
N-L{C}	4	821.4	829.4	840.81	829.73	834.04
TEE	2	825.19	829.19	834.89	829.29	831.51
EL	3	822.44	828.44	836.9961	828.6335	831.9164
EE	2	826.16	830.16	835.86	830.26	832.48

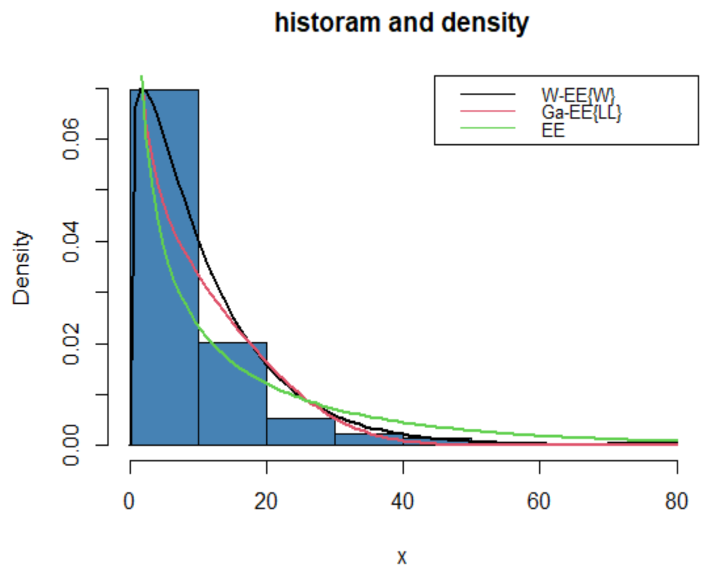

Figure 5: Histogram and density plot for the bladder cancer patient data set

6.2. Breaking strength of carbon fibre data

The second data set is uncensored data from [20] containing 100 observations on the breaking strength of carbon fibre (in Gba). The data set is as follows

3.70	2.74	2.73	2.50	3.60	3.11	3.27	2.87	1.47	3.11	4.40	2.41	3.19	3.22
1.69	3.28	3.09	1.87	3.15	4.90	3.75	2.43	2.95	2.97	3.39	2.96	2.53	2.67
2.93	3.22	3.39	2.81	4.20	3.33	2.55	3.31	3.31	2.85	2.56	3.56	3.15	2.35
2.55	2.59	2.38	2.81	2.77	2.17	2.83	1.92	1.41	3.68	2.97	1.36	0.98	2.76
4.91	3.68	1.84	1.59	3.19	1.57	0.81	5.56	1.73	1.59	2.00	1.22	1.12	1.71
2.17	1.17	5.08	2.48	1.18	3.51	2.17	1.69	1.25	4.38	1.84	0.39	3.68	2.48
0.85	1.61	2.79	4.70	2.03	1.80	1.57	1.08	2.03	1.61	2.12	1.89	2.88	2.82
2.05	3.65												

Table 6 summarises the MLEs and their respective standard error along with 2logLog for the proposed members of the “T-EE{Y}class” of probability distributions. To assess the applicability of the members of “T-EE{Y}class” of distributions, i.e., three-parameter W-EE{W}, four-parameter Ga-EE{LL}and N-EE{L} distributions are listed in Table 7. For the application of carbon fiber data set as compared to other existing distribution included Gamma Exponentiated Exponential (GEE), exponentiated exponential (EE) distribution, Kumaraswamy generalized Exponentiated Exponential (Kw-GEE) distribution, Transmuted Exponentiated Exponential (TEE)distribution, some goodness of test statistics including AIC, BIC, AICC, HQIC have been obtained. Figure 7 shows that our proposed three members of the class of distributions have the lowest value of all the test statistics. Therefore, the proposed three members named W-EE{W}, Ga-EE{LL} and N-EE{L} perform better than the existing ones. Figure 6 shows the comparative density plot of carbon fibre data.

Table 6:
MLEs and their Standard error and -2logL of carbon fibre data

Distributions	**Estimates (standard error)**								
	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{c}$	$\hat{a}$	$\hat{b}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\mu}$	$\hat{\sigma}$
W-EE{W}	1.6644 (1.2412)	0.4876 (0.2292)	0.4660 (0.1725)	-	-	1.000	-	-	-
W-EE{LL}	5.5128 (3.9134)	0.7182 (0.2681)	0.8360 (0.2681)	-	1.000	1.000	-	-	-
Ga - EE{LL}	0.2346 (0.1354)	0.0553 (0.1351)	-	1.00	0.4262 (0.6046)	3.5432 (5.4086)	1.0	-	-
N-EE{L}	0.3080 (2.200)	0.6447 (1.3380)	-	0.00	1.000	-	-	2.6925 (3.9213)	0.7712 (1.3734)

Table 7:
The statistics AIC, BIC, AICC and HQIC of carbon fibre data

Distribution	k	-2logL	AIC	BIC	AICC	HQIC
W-EE{W}	3	282.71	288.71	296.52	288.96	291.87
Ga-EE{LL}	4	282.72	290.72	301.14	291.14	294.94
N-EE{L}	4	282.67	290.67	301.09	291.09	294.89
GEE	3	287.48	293.48	301.30	293.73	296.64
EE	2	292.37	296.37	301.60	296.49	298.47
Kw-GEE	4	292.2	300.2	311.61	300.53	304.84
TEE	2	291.15	295.15	300.85	295.24	297.47

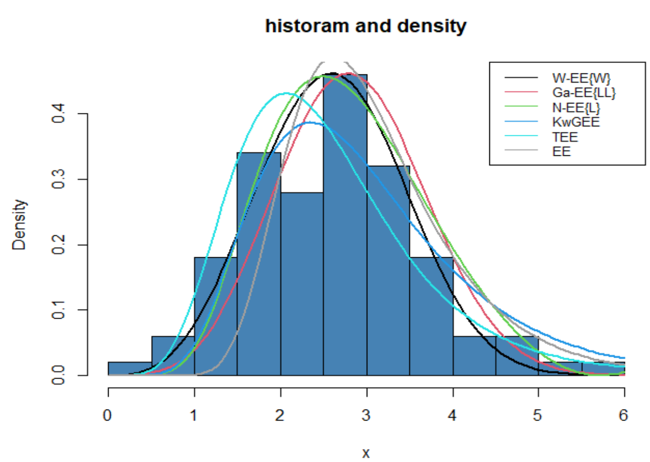

Figure 6: Histogram and density plot for the carbon fibre data

6.3. Failure time

The third data set is the failure time of 20 components also used by [21].

0.072	4.763	8.663	12.089	0.477	5.284	9.511	13.036	1.592	7.709	10.636
13.949	2.475	7.867	10.729	16.169	3.597	8.661	11.501	19.809		

The estimated values of parameters and their standard errors (SE) are shown in Table 8. Table 9 presents the model goodness of fit criteria, such as AIC, BIC, AICC, HQIC. We have compared the proposed members of the class of distributions with Gamma exponentiated exponential (GEE), beta exponential (BE), and exponentiated exponential (EE) distributions for the failure data. The figures in Table 9 show that members of the "T-EE{Y}class of distributions" are more applicable as compared to other existing distributions and Figure 7 shows the comparative density plot on failure time of 20-component data.

Table 8:
MLEs, Standard error and -2logL for failure data set

Distributions	**Estimates (standard error)**								
	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{c}$	$\hat{a}$	$\hat{b}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\mu}$	$\hat{\sigma}$
W-EE{W}	0.2547 (0.2098)	0.0195 (0.0317)	0.2741 (0.192)	-	-	1.000	-	-	-
W-EE{LL}	0.5646 (0.5766)	0.0368 (0.0558)	1.5949 (1.281)	-	1.000	1.000	-	-	-
Ga -EE{LL}	0.3970	0.0117	-	1.00	0.1621 (0.039)	0.2626 (0.1992)	1.00	-	-
N-EE{L}	3.0111 (0.1916)	1.4742 (0.0936)	-	-	-	-	-	10.9399 (2.0163)	8.3292 (1.396)

Table 9:
The statistics AIC, BIC, AICC and HQIC failure time of 20 components data

Distribution	**k**	**-2logL**	**AIC**	**BIC**	**AICC**	**HQIC**
W-EE{LL}	**3**	119.21	125.21	128.20	126.71	125.79
Ga-EE{LL}	**4**	117.49	125.49	129.48	128.16	126.27
W-EE{W}	**3**	120.65	126.65	129.64	128.15	127.23
TEED	**2**	125.22	129.22	134.92	129.32	131.54
GEE	3	62.345	130.69	133.68	132.19	131.27
BE	3	62.296	130.59	133.58	132.09	131.18
EE	2	62.384	128.77	130.76	129.48	129.16

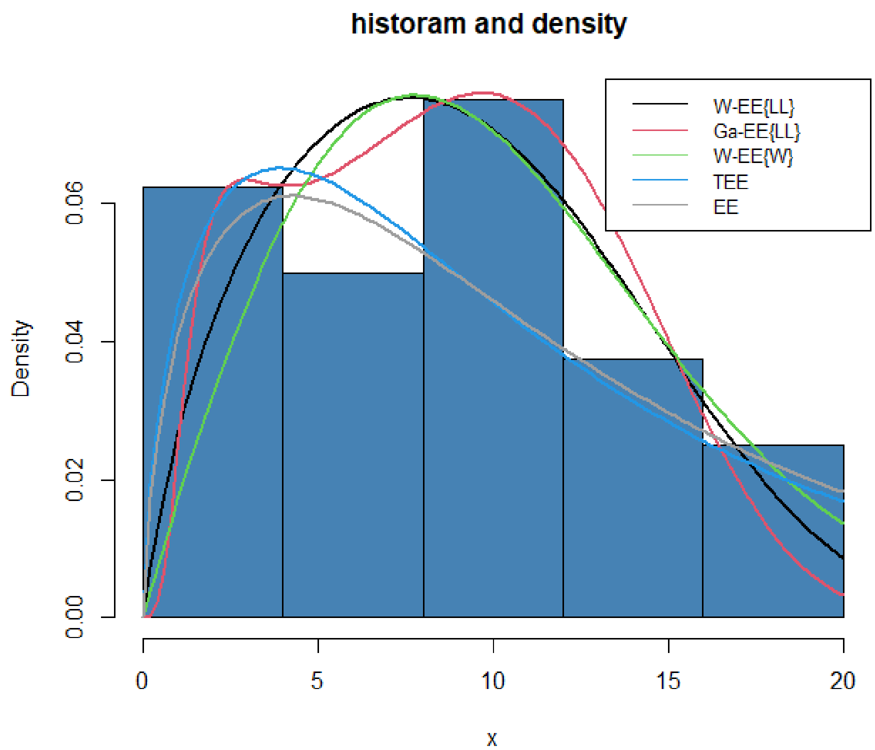

Figure 7: histogram and density plot for the failure time, data set 3

7. Conclusion

This research introduces a novel generalisation of the EED, named the T-EE{Y} class of probability distributions, which is constructed using the quantile functions of several well-known continuous probability distributions. Various properties of the proposed generator, including the survival function, hazard function, mode, quantile functions, upper bounds for moments, mean deviation, and Shannon's entropy, have been derived. Six subclasses of the T-EE{Y} class have been methodically investigated, along with their statistical properties. Additionally, four specific members of the T-EE{Y} class have been analysed mathematical, and practically as well, their graphical representations are also provided. The T-EE{Y} class demonstrates significant flexibility, exhibiting shapes that are unimodal, bimodal, symmetrical, and both right- and left-skewed. The parameter estimation of the selected members is derived by the maximum likelihood estimation (MLE) method. To evaluate the performance of the MLE estimators, a simulation study is conducted across various sample sizes. The practical applications of the proposed class are demonstrated through four specific members—W-EE{W}, W-EE{LL}, Ga-EE{W}, and N-EE{L}—by modelling them to three lifetime datasets and comparing their performance with other eminent probability distributions. Results demonstrate that the proposed members of the T-EE{Y} class outperformed several existing classes and other well-known distributions, highlighting their outstanding performance and practical utility.

Declaration

Author Contribution: conceptualization, theoretical and mathematical framework by S.B. & RI.; software handling, data visualization, and analysis; writing the first draft writing by S.B & R.I.; Concept, supervision, editing and reviewing the final draft by S.B.; Edited, reviewed and proofread the final draft, project management, and resources by S.B. All authors reviewed the final draft of the manuscript

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References

1. Gupta, R. C., Gupta, P. L. and Gupta, R. D. Modeling failure time data by Lehman alternatives. Communications in Statistics - Theory and Methods, 27(4), 887–904 (1998). https://doi.org/10.1080/03610929808832134
2. Masoom, A., Manisha, P. and Jungsoo, W. Some Exponentiated Distributions. The Korean Communications in Statistics, 14(1), 93-109 (2007).
3. Nadarajah, S. The exponentiated exponential distribution: a survey. AStA Adv Stat Anal 95, 219–251 (2011). https://doi.org/10.1007/s10182-011-0154-5
4. Abu-Youssef, S. E., Mohammed, B. I. and Sief, M. G. An Extended Exponentiated Exponential Distribution and its Properties. International Journal of Computer Applications, 121(5), 1–6 (2015). https://doi.org/10.5120/21533-4518
5. Hansain S. A., Iqbal, Z. and Ahmad M. On Exponentiated moment exponential distribution, Pakistan Journal of Statistics 31(2):267-280, (2015).
6. Al-Sulami, D. Exponentiated Exponential Weibull Distribution: Mathematical Properties and Application. American Journal of Applied Sciences, 17(1), 188–195 (2020). https://doi.org/10.3844/ajassp.2020.188.195
7. Alzaatreh, A., Famoye, F. and Lee, C. The gamma-normal distribution: Properties and applications. Computational Statistics &Amp; Data Analysis, 69, 67–80 (2014). https://doi.org/10.1016/j.csda.2013.07.035
8. Eugene, N., Lee, C. and Famoye, F. Beta-Normal Distribution and Its Applications. Communications in Statistics - Theory and Methods, 31(4), 497–512 (2002). https://doi.org/10.1081/sta-120003130
9. Alzaatreh, A., Lee, C. and Famoye, F. A new method for generating families of continuous distributions. METRON, 71(1), 63–79 (2013).
10. Cordeiro, G. M., Alizadeh, M. and Ortega, E. M. M. The Exponentiated Half-Logistic Family of Distributions: Properties and Applications. Journal of Probability and Statistics, 1–21 (2014). https://doi.org/10.1155/2014/864396
11. Oluyede, B. The Gamma-Weibull-G Family of Distributions with Applications. Austrian Journal of Statistics, 47(1), 45–76 (2018). https://doi.org/10.17713/ajs.v47i1.155
12. Tahir, M. H., Cordeiro, G. M., Alzaatreh, A., Mansoor, M., & Zubair, M. The logistic-X family of distributions and its applications, Communications in Statistics - Theory and Methods, 45:24, 7326-7349 (2016). DOI: 10.1080/03610926.2014.980516

13. Balogun, O. S. and Ieren, T. G. Exponential-Lindley Distribution: Theory and Application to Bladder Cancer Data, Journal of Applied Probability and Statistics, 16(2), 129-146 (2021).
14. Aljarrah, M. A., Lee, C. and Famoye, F. On generating T-X family of distributions using quantile functions. Journal of Statistical Distributions and Applications, 1(1) (2014). https://doi.org/10.1186/2195-5832-1-2
15. Ahmad, Z., Elgarhy, M. and Hamedani, G. G. A new Weibull-X family of distributions: properties, characterizations, and applications. Journal of Statistical Distributions and Applications, 5(1) (2018). https://doi.org/10.1186/s40488-018-0087-6
16. Alzaghal, A. and Hamed, D. New Families of Generalized Lomax Distributions: Properties and Applications. International Journal of Statistics and Probability, 8(6), 51 (2019). https://doi.org/10.5539/ijsp.v8n6p51
17. Nofal, Z. M., Altun, E., Afify, A. Z., and Ahsanullah, M. The Generalized Kumaraswamy-G Family of Distributions, Journal of Statistical Theory and Applications, 18(4), 329-342 (2018). : https://doi.org/10.2991/jsta.d.191030.001
18. Hamed D., Famoye, F. and Lee, C. On Families of Generalized Pareto Distributions: Properties and Applications, Journal of Data Science, 16(2), (2022). https://doi.org/10.6339/JDS.201804_16(2).0008
19. Hamed, D., and Alzaghal, A. New class of Lindley distributions: properties and applications. Journal of Statistical Distributions and Applications, 8, 11 (2021). https://doi.org/10.1186/s40488-021-00127-y
20. Nichols, M. D., and W. J. Padgett. 2006. A bootstrap control chart for Weibull percentiles. Quality and Reliability Engineering International 22 (2):141–51.
21. El-Gohary, A., & Abd El-Monsef, M. (2021). A New Extended Model with Bathtub-Shaped Failure Rate: Properties, Inference, Simulation, and Applications. Mathematics, 9(17), 2024. https://doi.org/10.3390/math9172024