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Sum of weighted gamma distribution: properties and applications in reliability engineering

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ABSTRACT

Applications of the lifetime continuous distributions in the reliability field are always in demand to improve the performance of electronic components. In this paper, we proposed a new method which is used to obtain a single-parameter lifetime distribution named "Sum of Weighted Gamma Distribution" (SWG distribution). The newly proposed distribution is obtained by weighing the gamma distribution with varying shape and constant scale parameters. The idea has been taken from the formation of the Lindley distribution, which is a mixture of exponential (with scale parameter θ) and gamma (with shape parameter 2 and scale parameter θ) distributions. Various mathematical properties of the SWG distribution have been derived. A few reliability and inequality measures, such as survival function, hazard rate, reversed hazard rate, cumulative hazard rate, Ginni indices, Lorenz and Bonferroni inequalities have been developed. Order statistics and upper record values from the SW-Gamma distribution have been studied. The parameter is estimated by using the method of maximum likelihood estimation (MLE), moreover, a simulation is conducted. Finally, the applications of the SWG distribution have been shown on three different lifetime data sets and compared with famous single-parameter lifetime distributions. It is shown that the SWG distribution is more flexible comparatively.

Keywords*:* Gamma distribution; SWG; MLE; record values; Lindley distribution; exponential distribution

1. Introduction

Exponential, Lindley, and many other single-parameter continuous distributions are available for modelling lifetime data sets. The exponential, the Lindley, the gamma, and the Weibull distributions are more common among these. However exponential and Lindley distributions both have one parameter, but the Lindley distribution is better as compared to the exponential distribution in the context of the hazard function, as the exponential distribution has a constant hazard function while the Lindley has a monotonically decreased hazard function.

Lindley [1] introduced a single-parameter distribution named Lindley in the framework of Bayesian Statistics, as a falsification of fiduciary statistics. Lindley distribution is a mixture of an exponential distribution with a scale parameter θ and a gamma distribution with fixed shape parameter as 2 and a scale parameter θ with a mixing ratio $\frac{\theta}{\theta+1}$ *and* $\frac{1}{\theta+1}$ respectively. Lindley distribution has its probability density function (pdf) and cumulative distribution function (cdf) as

$$f(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}; \qquad x > 0, \theta > 0$$

$$F(x;\theta) = 1 - \left(1 + \frac{\theta x}{\theta+1}\right)e^{-\theta x}; \qquad x > 0, \theta > 0$$

Ghitney et al [2] presented a thorough work including statistical properties and applications of the Lindley distribution. They showed that the Lindley distribution is better than the exponential distribution in many aspects, such as the hazard function of the Lindley distribution has an increasing trend while the exponential's hazard rate is a constant; moreover, mean residual life of the Lindley distribution is a decreasing function of the random variable.

Afterwards, various works have been done on the Lindley distribution, and the researchers proposed many distributions containing a single parameter as well as two parameters and generalisations of the Lindley distributions for modelling lifetime data. Lindley distribution is widely used in biomedical, reliability engineering, behavioural sciences, and many other.

Shanker [3] proposed a new single-parameter distribution named Akash distribution by changing the mixing proportion of exponential and gamma, which is a modification of one one-parameter Lindley distribution. He applied it to biological data and revealed its flexibility over exponential and Lindley distributions. Shanker [4] developed another distribution by modifying the techniques and named it shanker distribution, Shanker et al [5] presented a detailed comparative study of the exponential and Lindley distributions to model lifetime dates sets and they concluded that in some cases Lindley is better than exponential and, in some cases, exponential is better than Lindley distribution. Shanker and Shukla [6] introduced the Ishita distribution, which is another distribution of modifies the mixing proportion. They used it to model in the biomedical and engineering fields and showed its superiority over the existing ones. Shukla [7] offered another single-parameter distribution named the Pranav distribution with its several essential properties and applications. Qayoom et al. [8] presented the DUS Lindley distribution by applying the DUS transformation on the classical Lindley distribution. Alzawq et al. [9] used the classical Lindley distribution to propose a new generalisation of the Lindley distribution named a logarithmic-Lindley (Log-L) distribution. Elgarhy et al. [10] proposed an extension of the classical Lindley distribution named as EX-Lindley model. Karakus et al. [11] introduced a unit power Lindley distribution with application for unit interval-based data.

In above mentioned distributions, all are modifying forms of the Lindley distribution by changing the mixing proportion of exponential and gamma distributions. In this research, we generalised the modification technique of this mixing proportion and developed a general mixing proportion which gives us a general form of all these distributions. This generalised mixing technique is considered by taking the gamma scale and shape parameters in a manner that is considered in previously mentioned distributions but instead of developing another same kind of distribution we presented the mixing proportion by weighing gamma distribution with varying shape parameter from 1 to 4, and scale parameter as $'\theta'$.

There are diverse circumstances where every proposed lifetime distribution might not be providing a better fit from either a hypothetical or practical point of view. Therefore, there is a need for the development of a distribution which can be used in such circumstances. There are several literatures which support the Lindley distribution in various kind of real-life situations by generalizing Lindely distribution in many ways such as: Hamed and Alzaghal [12] proposed a new class of Lindley distribution in which they developed a T-Lindley(Y) class of distributions using the quantile function of various well know distributions. Algarni [13] proposed a new generalisation of the Lindley distribution using the Marshall-Olkin method and its submodels are presented. But these are the distributions which have an enormous number of parameters and complex expressions of pdfs and cdfs. Alrasheedi et al. [14] proposed a new extension of the Lindley distribution with applications. Ahsan-ul-Haq et al. [15] proposed a new generalisation of the Lindley distribution for modelling wind speed, Al-Nuaami et al. [16] introduced the Poisson-Lindley distribution with applications to statistical process control.

This paper introduces a novel single-parameter lifetime distribution, termed the Sum of Weighted Gamma (SWG) distribution, derived by weighting gamma distributions with varying shape and constant scale parameters. The formulation is inspired by the Lindley distribution, which is a mixture of exponential and gamma distributions. We comprehensively explore the mathematical properties of the SWG distribution, including its reliability functions (such as the survival function, hazard rate, and cumulative hazard rate) and inequality measures (including Gini indices, Lorenz, and Bonferroni curves). The study further examines order statistics and upper record values from the SWG distribution. The model parameters are estimated using the maximum likelihood estimation method, supported by a simulation study. The practical utility of the distribution is demonstrated through applications to three real lifetime datasets, where the SWG distribution shows superior flexibility and performance when compared to existing single-parameter lifetime models.

The novelty of this work lies in the development of a new single-parameter lifetime distribution, the Sum of Weighted Gamma (SWG) distribution, constructed through a unique weighting approach of gamma distributions, a concept not previously applied in this context. Unlike traditional gamma-based models, the SWG distribution retains analytical tractability while offering enhanced modelling flexibility. This is the first instance where such a weighting mechanism has been employed to construct a continuous lifetime distribution, along with the derivation of its comprehensive reliability and inequality properties, and its application validated through real-world data comparison with other well-known models.

Therefore, there is scope for searching new lifetime distributions which are simpler in expression as well as have a smaller number of parameters which can be flexible than the Lindley type distributions, such as Akash, Shankar, Lindley, Exponential, Pranav and Sujata distributions. In the aforesaid distributions, they used the two-component or three-component mixture of exponential and gamma distributions. Herein this article, we are generalising the selection of the mixing proportion. Therefore, in section 2, we introduce a new method to obtain a one-parameter probability distribution along with a graphical representation. In section 3, some statistical properties of the proposed model are developed and in section 4, reliability and inequality measures for the SWG distribution are presented. In section 5, order statistics, in section 6, parameter estimation and simulations are presented. Finally, the applications of the SWG distribution are presented in section 7, it is shown that the proposed model is more flexible as compared to the Lindley type distributions. In section 8, the conclusion is discussed.

2. Formulation of the Proposed Model

In this paper, we presented a new single-parameter distribution for modelling lifetime data. The distribution is obtained by weighing the gamma distribution with varying shape parameter from 1 to 4, and scale parameter as $'\theta'$. This pdf is a summation of different Gamma distributions, with assigning a proportion of weights to each Gamma pdf. Gamma distributions have been put together by giving different weights as $\theta^4, \theta^2, 2\theta, 6$ respectively.

2.1. Methodology

The idea of the selection of the weights has been taken from the development of the Lindley, Shankar, Parnav, Akash, and Sujatha distributions. They have taken these weights as two-component and three-component mixing proportions, by changing the gamma scale and shape parameters along with the exponential distribution. They have taken one fixed value for the gamma shape parameter, e.g., 2 or 3, along with an unknown scale parameter for two-component mixing proportion, and three-component mixing proportion, they have taken two fixed values of the gamma shape parameter along with an unknown scale parameter. But we have taken 4 fixed values for the gamma shape parameter along with an unknown scale parameter, by changing the weights for every set of gamma parameters. Table 1 shows the proposed weights and related gamma pdfs.

Table 1:
Some proposed weights and related gamma pdfs to obtain the SWG model

Weights	Shape and scale parameters	Gamma PDFS with
$\boldsymbol{w_1 = \theta^4}$	Gamma $(1,\theta)$	$f_1(x) = \theta e^{-\theta x}$
$\boldsymbol{w_2 = \theta^2}$	Gamma $(2,\theta)$	`$f_2(x) = \theta^2 x e^{-\theta x}$
$\boldsymbol{w_3 = 2\theta}$	Gamma $(3,\theta)$	$f_3(x) = \frac{\theta^3 x^2 e^{-\theta x}}{2}$
$\boldsymbol{w_4 = 6}$	Gamma $(4,\theta)$	$f_4(x) = \frac{\theta^4 x^3 e^{-\theta x}}{6}$

The following is the proposed methodology to obtain the sum of weighted gamma distribution (SWG distribution) is given as

$$f(x;\theta) = \frac{\sum_{i=1}^{4} w_i\, f_i(x)}{\sum_{i=1}^{4} w_i}$$

OR

$$f(x;\theta) = \frac{w_1 f_1(x) + w_2 f_2(x) + w_3 f_3(x) + w_4 f_4(x)}{\sum_{i=1}^{4} w_i} \qquad (1)$$

Where, $\sum_{i=1}^{4} w_i = w_1 + w_2 + w_3 + w_4$

Using the weight and pdfs given in Table 1, and substituting these in the equ. (1), after some simplifications we get the pdf of the newly proposed distribution with one parameter θ as

$$f(x,\theta) = \frac{\theta^4}{\theta^4+\theta^2+2\theta+6}(\theta + x + x^2 + x^3)e^{-\theta x}; \qquad x > 0, \theta > 0 \tag{2}$$

Where the corresponding cumulative distribution function of the random variable X is given as

$$F_X(x) = P(X \leq x)$$

$$F(x;\theta) = 1 - \left(1 + \frac{\theta^3 x^3 + (\theta^3+3\theta^2)x^2 + (\theta^3+2\theta^2+6\theta)x)}{\theta^4+\theta^2+2\theta+6}\right) e^{-\theta x} \; ; x > 0, \theta > 0 \tag{3}$$

Moreover, in the eq. (1), by using different proportions of weights and gamma PDFs, we can get various single-parameter continuous distributions. The PDF given eq. (1) is a generator to develop various continuous probability distributions by changing weights in it.

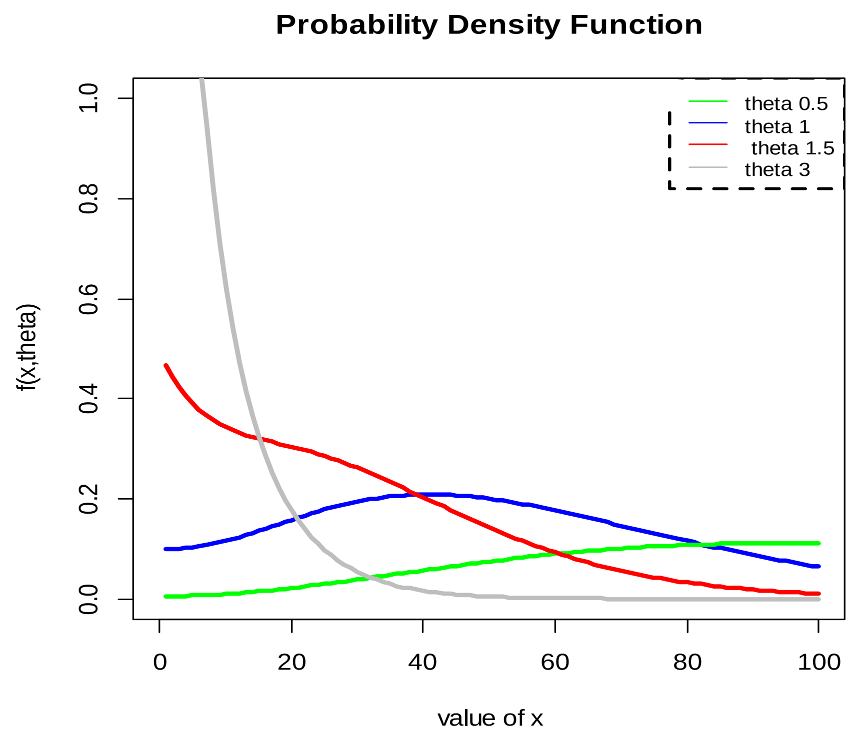

Figure 1: Density plot for SWG distribution

From Figure 1, it is observed that SWG distribution shows a variety of shapes as left-skewed, right-skewed, and symmetric, for various values of θ.

3. Some Structural Properties of SWG Distribution

Let the random variable (RV) 'X' follow the SW-Gamma distribution with parameter θ given in eq. (2), then the rth raw moments have been obtained as

$$\acute{\mu}_r = E(X^r) = \int_0^\infty x^r f(x;\theta)dx$$

$$\acute{\mu}_r = \frac{\theta^4}{\theta^4+\theta^2+2\theta+6}\left[\frac{\Gamma(r+1)}{\theta^r}+\frac{\Gamma(r+2)}{\theta^{r+2}}+\frac{\Gamma(r+3)}{\theta^{r+3}}+\frac{\Gamma(r+4)}{\theta^{r+4}}\right]; r = 1,2,\ldots \quad (4)$$

The first four raw moments, including mean, are given below:

Mean = $\acute{\mu}_1 = \frac{\theta^4+2\theta^2+6\theta+24}{\theta(\theta^4+\theta^2+2\theta+6)}$, $\acute{\mu}_2 = \frac{2\theta^4+6\theta^2+24\theta+120}{\theta^2(\theta^4+\theta^2+2\theta+6)}$, $\acute{\mu}_3 = \frac{6\theta^4+24\theta^2+120\theta+720}{\theta^3(\theta^4+\theta^2+2\theta+6)}$, $\acute{\mu}_4 = \frac{24\theta^4+120\theta^2+720\theta+5040}{\theta^4(\theta^4+\theta^2+2\theta+6)}$, rth central moments are given as

$$\mu_k = E(x-\mu)^k = (-1)^{k-r}\sum_{r=0}^{k}\binom{k}{r}\acute{\mu}_r(\mu)^{k-r} \quad (5)$$

The central moments of the SWG distribution are given as

$$Mean = \acute{\mu}_1 = \frac{\theta^4+2\theta^2+6\theta+24}{\theta(\theta^4+\theta^2+2\theta+6)} = \mu \quad (6)$$

The variance of the SWG distribution is denoted is as given

$$\mu_2 = \frac{\theta^8+4\theta^6+16\theta^5+86\theta^4+12\theta^3+72\theta^2+96\theta+144}{\theta^2(\theta^4+\theta^2+2\theta+6)^2} = \sigma^2 \quad (7)$$

$$\mu_3 = \frac{2\theta^{12}+12\theta^{10}+60\theta^9+2928\theta^8+84\theta^7+3856\theta^6+2412\theta^5-127800\theta^4+34432\theta^3-60738\theta^2-129600\theta+2082240}{\theta^3(\theta^4+\theta^2+2\theta+6)^3} \quad (8)$$

$$\mu_4 = \frac{1}{\theta^4(\theta^4+\theta^2+2\theta+6)^4}[9\theta^{16}+72\theta^{14}+384\theta^{13}+2940\theta^{12}+1032\theta^{11}+8304\theta^{10}+13344\theta^9+291112\theta^8+247768\theta^7+652232\theta^6+924448\theta^5+1300032\theta^4+77184\theta^3+1313328\theta^2++1244416\theta+93312] \quad (9)$$

The coefficient of variation of the SWG distribution

$$C.V = \frac{\sigma}{\mu} = \frac{[\theta^8+4\theta^6+16\theta^5+86\theta^4+12\theta^3+72\theta^2+96\theta+144]^{1/2}}{(\theta^4+2\theta^2+6\theta+24)} \quad (10)$$

The coefficient of skewness is

$$\sqrt{\beta_1} = \frac{\left[\begin{matrix}2\theta^{12}+12\theta^{10}+60\theta^9+2928\theta^8+84\theta^7+3856\theta^6+2412\theta^5-127800\theta^4+34432\theta^3-\\ 60738\theta^2-129600\theta+2082240\end{matrix}\right]}{[\theta^8+4\theta^6+16\theta^5+86\theta^4+12\theta^3+72\theta^2+96\theta+144]} \quad (11)$$

The coefficient of kurtosis is

$$\beta_2 = \frac{\left[\begin{array}{c} 9\theta^{16}+72\theta^{14}+384\theta^{13}+2940\theta^{12}+1032\theta^{11}+8304\theta^{10}+13344\theta^{9}+29112\theta^{8}+24768\theta^{7}+65232\theta^{6}+92448\theta^{5}+ \\ 130032\theta^{4}+77184\theta^{3}+131328\theta^{2}++124416\theta+93312 \end{array}\right]}{\left[\theta^{8}+4\theta^{6}+16\theta^{5}+86\theta^{4}+12\theta^{3}+72\theta^{2}+96\theta+144\right]} \tag{12}$$

The moment generating function of the SWG distribution is

$$M_x(t) = \frac{\theta^4}{\theta^4+\theta^2+2\theta+6} \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(\frac{\Gamma(k+1)}{\theta^k} + \frac{\Gamma(k+2)}{\theta^{k+2}} + \frac{\Gamma(k+3)}{\theta^{k+3}} + \frac{\Gamma(k+4)}{\theta^{k+4}} \right) \tag{13}$$

It can also be written in the form:

$$M_x(t) = \sum_{t=0}^{\infty} \frac{t^r}{r!} \acute{\mu}_r \tag{14}$$

Incomplete moments of a distribution form a natural building block for measuring inequality e.g., the Lorenz curve, Bonferroni curve and Gini measures of inequality. Let X follow the SWG distribution having parameter θ. The nth incomplete moment is given as

$$I_X(t) = \int_0^t x^n f(x) dx\,; \qquad n = 1,2,\ldots$$

$$= \frac{\theta^4}{\theta^4+\theta^2+2\theta+6} \left[\frac{\theta\Gamma((r+1),0)-\Gamma((r+1),\theta x)}{\theta^{r+1}} + \frac{\Gamma((r+2),0)-\Gamma((r+2),\theta x)}{\theta^{r+2}} + \frac{\Gamma((r+3),0)-\Gamma((r+3),\theta x)}{\theta^{r+3}} + \frac{\Gamma((r+4),0)-\Gamma((r+4),\theta x)}{\theta^{r+4}} \right] \quad ()j$$

$$I_X(t) = \frac{\theta^4}{\theta^4+\theta^2+2\theta+6} \left[\frac{\theta\Gamma((r+1),0)-\Gamma((r+1),\theta x)}{\theta^{r+1}} + \sum_{j=2}^{4} \frac{\Gamma((r+j),0)-\Gamma((r+j),\theta x)}{\theta^{r+j}} \right] \tag{15}$$

Where $\Gamma(r,t) = \int_t^{\infty} x^{r-1} e^{-x} dx$ is the upper incomplete gamma function. For $r = 1$, we get the 1st incomplete moment for the pdf in equ. (2)

$$\int_0^t x f(x) dx = \mu - e^{-\theta t} \left(\mu + \mu\theta t + \frac{(\theta^4 t^4 + (\theta^4+4\theta^3)t^3 + (\theta^4+3\theta^3+12\theta^2)t^2)}{\theta(\theta^4+\theta^2+2\theta+6)} \right) \tag{16}$$

1st incomplete moment is also used in mean deviation (M.D) computation. M.D measures the amount of dispersion in the population.

$$\delta_1(X) = \int_0^{\infty} |x-\mu| f(x) dx,$$

and

$$\delta_2(X) = \int_0^{\infty} |x-M| f(x) dx$$

Where $\mu = mean$ and M =median of the distribution. M.D about mean is:

$$\delta_1(X) = E\{|X-\mu|\} = \int_0^\infty |x-\mu| f(x)dx$$

$$\delta_1(X) = 2e^{-\theta\mu}\left[\mu + \frac{\{\theta^3\mu^3+(\theta^3+6\theta^2)\mu^2+(\theta^3+4\theta^2+18\theta)\mu\}}{\theta(\theta^4+\theta^2+2\theta+6)}\right] \quad (17)$$

M.D about median is

$$\delta_2(X) = \int_0^\infty |x-m| f(x)dx$$

$$\delta_2(X) = 2e^{-\theta m}\left[\mu + \frac{\{\theta^3 m^3+(\theta^3+6\theta^2)m^2+(\theta^3+4\theta^2+18\theta)m\}}{\theta(\theta^4+\theta^2+2\theta+6)}\right] \quad (18)$$

Shannon entropy of SWG distribution is

$$H(x) = -\int_0^\infty f(x) ln f(x)dx$$

$$H(X) = -ln\theta^4 + \theta\mu + ln(\theta^4+\theta^2+2\theta+6) - \frac{\theta^4}{\theta^4+\theta^2+2\theta+6}\psi_x(\theta) \quad (19)$$

Where,

$$\psi_x(\theta) = \int_0^\infty (\theta + x + x^2 + x^3)e^{-\theta x} ln(\theta + x + x^2 + x^3)dx \quad \text{jj}$$

4. Reliability and Inequality Measures

In this section, we established the Lorenz (L) and Bonferroni (B) curves, Gini and Bonferroni indices and some reliability properties like the reliability (survival) function, hazard function (HF), reversed hazard function (RHF), and cumulative hazard function (CHF). L and B curves are graphical elucidations of the degree of inequality of distribution for a random variable in the case of economic/wealth conditions of a nation/country. These curves have extensive applications in many fields such as reliability engineering, economy, biomedicine, and insurance.

The L and B curves for a random variable X are defined as

$$B(p) = \frac{1}{p\mu}\int_0^t xf(x)dx = \frac{1}{p\mu}\left[\mu - \int_t^\infty xf(x)dx\right] \quad (20)$$

and

$$L(p) = \frac{1}{\mu}\int_0^t xf(x)dx = \frac{1}{\mu}\left[\mu - \int_t^\infty xf(x)dx\right] \qquad (21)$$

and equivalently

$$B(p) = \frac{1}{p\mu}\int_0^p F^{-1}(x)dx$$

and

$$L(p) = \frac{1}{\mu}\int_0^p F^{-1}(x)dx$$

The Bonferroni and Gini indices are as follows:

$$B = 1 - \int_0^1 B(p)dp \qquad (22)$$

$$G = 1 - 2\int_0^1 L(p)dp \qquad (23)$$

Using eq (16) in eq (20) and (21), we get

$$B(p) = \frac{1}{p}\left[1 - e^{-\theta t}\left(1 + \theta t + \frac{\theta^4 t^4 + (\theta^4 + 4\theta^3)t^3 + (\theta^4 + 3\theta^2 + 12\theta^2)t^2}{(\theta^4 + 2\theta^2 + 6\theta + 24)}\right)\right] \qquad (24)$$

and

$$L(p) = \left[1 - e^{-\theta t}\left(1 + \theta t + \frac{\theta^4 t^4 + (\theta^4 + 4\theta^3)t^3 + (\theta^4 + 3\theta^2 + 12\theta^2)t^2}{(\theta^4 + 2\theta^2 + 6\theta + 24)}\right)\right] \qquad (25)$$

Using eq (24) and (25) in (22) and (23), Gini and Bonferroni indices are obtained as

$$G = -1 + \frac{2\{(2\theta^4 + 6\theta^2 + 18\theta + 126) - e^{-\theta}(\theta^5 + 5\theta^4 + 15\theta^3 + 57\theta^2 + 120\theta + 126)\}}{\theta(\theta^4 + 2\theta^2 + 6\theta + 24)} \qquad (26a)$$

$$B = 1 - \frac{\left(\theta^5 - 2\theta^4 + 2\theta^3 + 6\theta - 126) + e^{-\theta}(\theta^5 + 5\theta^4 + 15\theta^3 + 57\theta^2 + 120\theta + 126)\right)}{p\theta(\theta^4 + 2\theta^2 + 6\theta + 24)} \qquad (26b)$$

The survivor (reliability) function is the probability that the event has not yet occurred by time. Stated differently, R(x) denotes the probability of surviving/reliability beyond a specific time. The reliability function of the SWG distribution is defined as

$$R(x) = \left[1 + \frac{\theta^3 x^3 + (\theta^3 + 3\theta^2)x^2 + (\theta^3 + 2\theta^2 + 6\theta)x)}{\theta^4 + \theta^2 + 2\theta + 6}\right]e^{-\theta x} \qquad (27)$$

The HF of SWG distribution is:

$$h(x)=\frac{\theta^4(\theta+x+x^2+x^3)}{[(\theta^4+\theta^2+2\theta+6)+\theta^3x^3+(\theta^3+3\theta^2)x^2+(\theta^3+2\theta^2+6\theta)x)]} \tag{28}$$

The CHF of the SWG distribution is

$$Z(x)=\theta x-ln\left[1+\frac{\theta^3x^3+(\theta^3+3\theta^2)x^2+(\theta^3+2\theta^2+6\theta)x)}{\theta^4+\theta^2+2\theta+6}\right] \tag{29}$$

The RHF of the SWG distribution is

$$r(x)=\frac{\theta^4(\theta+x+x^2+x^3)e^{-\theta x};}{(\theta^4+\theta^2+2\theta+6)\left[1-\left(1+\frac{\theta^3x^3+(\theta^3+3\theta^2)x^2+(\theta^3+2\theta^2+6\theta)x)}{\theta^4+\theta^2+2\theta+6}\right)e^{-\theta x}\right]} \tag{30}$$

Figure 2 displays the survival function of SWG distribution for various values of θ and obviously the shape is monotonically decreasing. From Figure 3, it can be seen that the hazard function of the SWG distribution is monotonically increasing for various values of θ. For $\theta=1$ it is increasing but in a very slow trend.

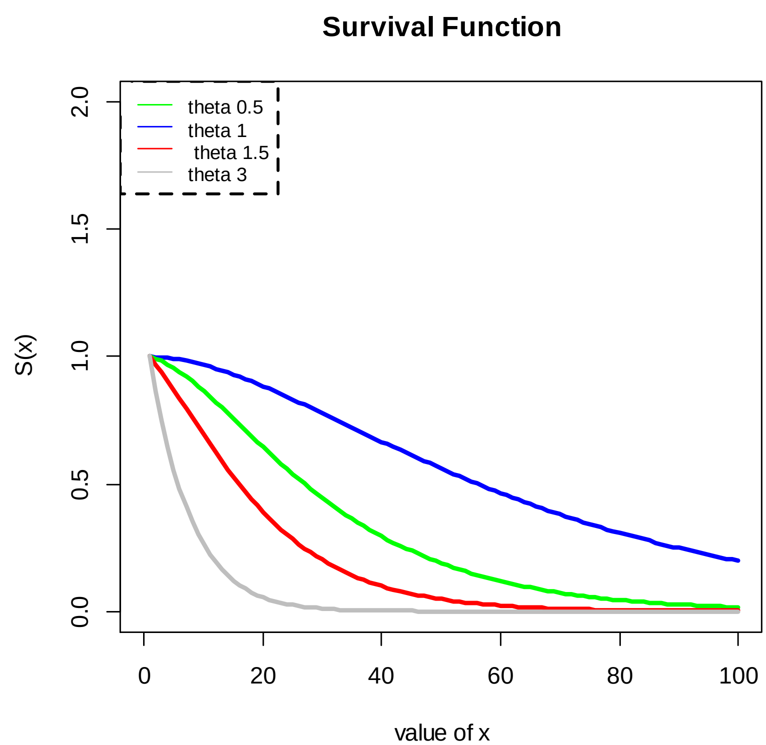

Figure 2: Graph of survival function

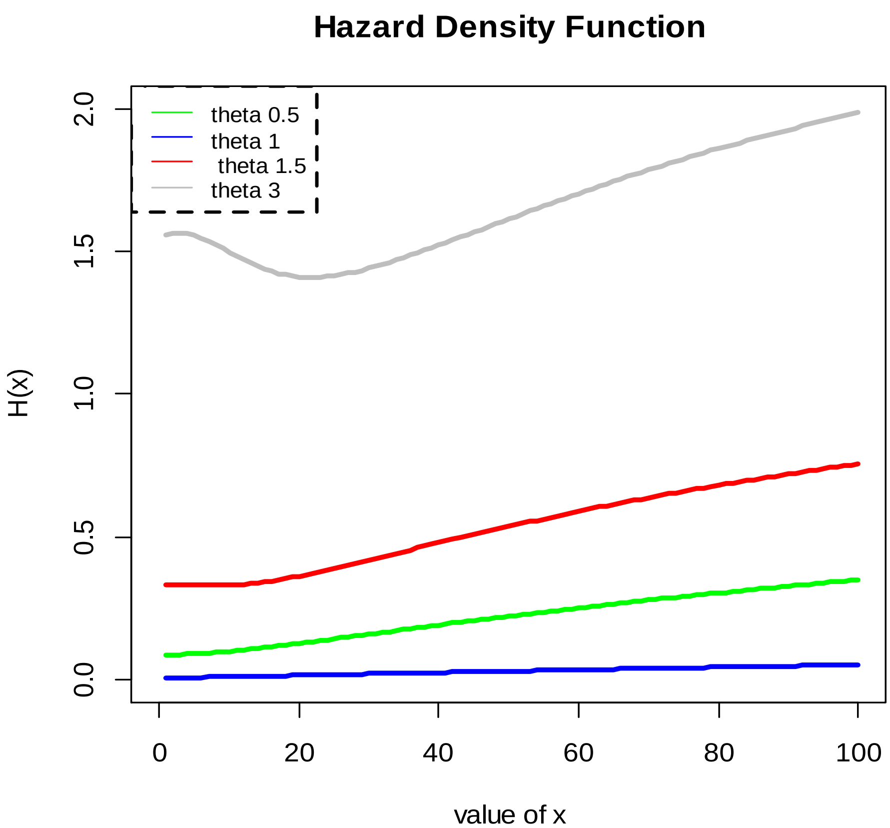

Figure 3: Graph of hazard function

5. Reliability and Inequality Measures

In this section, we presented the order statistics and upper record values theory from the SWG distribution.

Let $X_1, X_2, X_3, \ldots . X_n$ be a random sample from the SWG distribution with parameter θ, and $X_{(1)}, X_{(2)}, X_{(3)}, \ldots, X_{(n)}$ be their order statistics, then the cdf and pdf of their rth order statistics are

$$F_X(x) = \sum_{j=k}^{n} \binom{n}{j} e^{-n\theta x} \left[1 - \left(1 + \frac{\theta^3 x^3 + (\theta^3 + 3\theta^2) x^2 + (\theta^3 + 2\theta^2 + 6\theta) x)}{\theta^4 + \theta^2 + 2\theta + 6}\right) e^{-\theta x}\right]^j \left[\left(1 + \frac{\theta^3 x^3 + (\theta^3 + 3\theta^2) x^2 + (\theta^3 + 2\theta^2 + 6\theta) x)}{\theta^4 + \theta^2 + 2\theta + 6}\right) e^{-\theta x}\right]^{n-j} \quad (31)$$

$$f_{X_{(r)}}(x;\theta) = \frac{n!}{(n-r)!(r-1)!} \frac{\theta^4 (\theta + x + x^2 + x^3) e^{-\theta x}}{(\theta^4 + \theta^2 + 2\theta + 6)} \times \text{ k}$$

$$\left[1 - \left(1 + \frac{\theta^3 x^3 + (\theta^3 + 3\theta^2) x^2 + (\theta^3 + 2\theta^2 + 6\theta) x)}{\theta^4 + \theta^2 + 2\theta + 6}\right) e^{-\theta x}\right]^{r-1} \left[\left(1 + \frac{\theta^3 x^3 + (\theta^3 + 3\theta^2) x^2 + (\theta^3 + 2\theta^2 + 6\theta) x)}{\theta^4 + \theta^2 + 2\theta + 6}\right) e^{-\theta x}\right]^{n-r} \quad (32)$$

For $r = 1$, we get 1st order statistic of the SWG distribution:

$$f_{X_{(1)}}(x;\theta)=\frac{n\theta^4(\theta+x+x^2+x^3)e^{-\theta x}}{\theta^4+\theta^2+2\theta+6}\left[\left\{\left(1+\frac{\theta^3x^3+(\theta^3+3\theta^2)x^2+(\theta^3+2\theta^2+6\theta)x)}{\theta^4+\theta^2+2\theta+6}\right)e^{-\theta x}\right\}\right]^{n-1} \quad (33)$$

For $r=n$, we get nth order statistic of the SWG distribution:

$$f_{X_{(n)}}(x;\theta)=\frac{n\theta^4(\theta+x+x^2+x^3)e^{-\theta x}}{\theta^4+\theta^2+2\theta+6}\left[1-\left(1+\frac{\theta^3x^3+(\theta^3+3\theta^2)x^2+(\theta^3+2\theta^2+6\theta)x)}{\theta^4+\theta^2+2\theta+6}\right)e^{-\theta x}\right]^{n-1} \quad (34)$$

Let $X_1, X_2, X_3, \ldots . X_n$ be the random sample with arises from the SWG distribution and $X_{U(1)}, X_{U(2)}, X_{U(3)}, \ldots, X_{U(n)}$ be the upper record values, then the density function of the upper record value from the SWG distribution is obtained as

$$f_{X_{U(n)}}(x)=\frac{1}{\Gamma(n)}[-ln\{(\theta^4+\theta^2+2\theta+6)+\theta^3x^3+(\theta^3+3\theta^2)x^2+(\theta^3+2\theta^2+6\theta)x)\}+ ln(\theta^4+\theta^2+2\theta+6)-\theta x]^{n-1}\left[\frac{\theta^4(\theta+x+x^2+x^3)e^{-\theta x}}{\theta^4+\theta^2+2\theta+6}\right] \quad (35)$$

6. Maximum Likelihood Estimation (MLE)

The SW-gamma distribution is a single-parameter density which is a scale parameter. In this section, we presented MLE estimation to estimate the parameter of the SW-gamma distribution. The log likelihood function of the SWG distribution is:

$$lnl=4nln\theta-nln(\theta^4+\theta^2+2\theta+6)+\sum_{i=1}^{n}\ln(\theta+x_i+{x_i}^2+{x_i}^3)-\theta\sum_{i=1}^{n}x_i \quad (36)$$

Differentiate partially w.r.tθ, we have

$$\frac{\partial lnl}{\partial\theta}=\frac{n(2\theta^2+6\theta+24)}{\theta(\theta^4+\theta^2+2\theta+6)}+\sum_{i=1}^{n}\frac{1}{(\theta+x_i+{x_i}^2+{x_i}^3)}-n\bar{x} \quad (37)$$

6.1. Simulation Study

In this section, a Monte Carlo simulation study is conducted by taking small, medium and larger sample sizes from iterations of 10,000. The average biases, biases, mean square errors (MSE) and mean relative errors (MRF) are calculated and shown in the table below.

From Tables 2 & 3, it is observed that as the sample size increases from 20 to 100, the MSE is reducing but at some values of the theta it is not true, it might be concluded that the MSE is not good for large sample sizes particularly more than 100, for SWG distribution instead other estimation methods can be used if they perform better.

For this reason, the proposed density is applied to the data sets which have a sample size less than 300, and MLE is used to estimate the parameters.

Table 2:
Simulation results for MLE with n = 20, 50, 100 and 300

		20	**50**	**100**	**300**
θ = 0.05	Average bias	1.9421	1.7716	1.7769	1.9040
	Biases	1.8921	1.7216	1.7269	1.8540
	MSE	3.5802	2.9639	2.9823	3.4375
	MRE	37.8430	34.4318	34.5388	37.0810
θ = 1.5	Average bias	2.2092	1.8514	1.7764	2.6527
	Biases	0.7092	0.3514	0.2769	1.1527
	MSE	0.5030	0.1235	0.0767	1.3286
	MRE	0.4728	0.2343	0.1846	0.7684
θ = 2.0	Average bias	2.2092	1.8514	1.7769	2.6527
	Biases	0.2092	-0.1485	-0.2231	0.6527
	MSE	0.0438	0.0221	0.0498	0.4259
	MRE	0.1046	0.0742	0.1115	0.3263

Table 3:
Simulation results for MLE with n = 20, 50, 100 and 300

		20	**50**	**100**	**300**
θ = 0.5	Average bias	1.9421	1.8515	1.7769	1.9041
	Biases	1.4422	1.3515	1.2769	1.4041
	MSE	2.0798	1.8265	1.6306	1.9714
	MRE	2.8843	2.7029	2.5539	2.8081
θ = 1.2	Average bias	2.2092	1.8515	1.7769	2.6527
	Biases	1.0092	0.6515	0.5769	1.4527
	MSE	1.0186	0.4244	0.3329	2.1102
	MRE	0.8410	0.5429	0.4808	1.2105
θ = 2.5	Average bias	2.2092	1.8515	1.7769	2.6526
	Biases	-0.2908	-0.6485	-0.7231	0.1526
	MSE	0.0845	0.4206	0.5228	0.0233
	MRE	0.1163	0.2594	0.2892	0.0611

7. Applications

In this section, the proposed density, named the SWG distribution, is modelled on three lifetime data sets. The parameter of the SWG distribution is estimated by MLE with standard errors, -2 log likelihood, and goodness of fit tests. In three data sets, we compared SWG distribution with some famous single-parameter and Lindley-type distributions. Overall SWG distribution shows flexibility over the competitive densities.

- **Data Set 1:** The first data set represents the survival times (in days) of 72 guinea pigs affected with virulent tubercle bacilli, and it is reported by Bjerkedal [17].
- **Data Set 2:** The second data set is the strength data of the glass of the aircraft window, and it is reported by Fuller et al. [18].
- **Data Set 3:** The third data set represent the tensile strength, measured in GPa, of 69 carbon fibres tested under tension at gauge lengths of 20 mm and it is considered from Bader and Priest [19].

Table 4:
MLE estimates and statistics of the distribution of survival times (ib days) of guinea pigs

Distributions	Estimates	Standard Error	-2logL	AIC	AICC	BIC
SWG Distribution	1.46899	0.07588558	210.837	216.7476	216.8047	219.0243
Parnav Distribution	1.478	0.0689	225.0703	227.0703	227.1274	226.9276
Akash Distribution	1.2227	0.0815	216.1373	218.1373	218.1947	220.414
Ishita Distribution	0.8847	0.0592	219.224	221.224	221.3011	221.1013
Lindley Distribution	1.8744	0.0771	227.051	229.051	229.1081	228.9083

Table 5:
MLE estimates and statistics of distributions of strength of glass of the aircraft window

Distributions	Estimates	-2logL	AIC	AICC	BIC
SWG Distribution	0.1283209	233.3929	235.3034	235.3605	237.5801
Sujatha Distribution	0.09561	241.50	243.50	243.64	244.94
Akash Distribution	0.09762	240.68	242.68	242.82	244.11
Shankar Distribution	0.064712	254.35	254.35	254.49	255.78
Lindley Distribution	0.062988	253.99	255.99	256.13	257.42
Exponential Distribution	0.032455	274.53	276.53	276.67	277.96

Table 6:
MLE estimates and statistics of distributions of tensile strength in GPA of 69 carbon Fibres

Distributions	Estimates	-2logL	AIC	AICC	BIC
SWG Distribution	***1.194522***	***211.0481***	***213.0481***	***213.1052***	***215.3248***
Sujatha Distribution	0.936119	221.61	223.61	225.84	225.84
Akash Distribution	0.964726	224.28	226.34	228.51	228.51
Shankar Distribution	0.658029	233.01	235.06	237.24	237.24
Lindley Distribution	0.659000	238.38	240.44	240.44	242.61
Exponential Distribution	0.407941	261.74	263.74	263.80	265.97

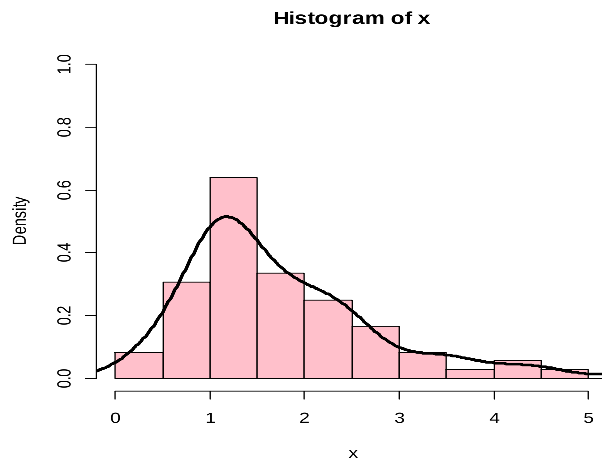

Figure 4: histogram for data 1

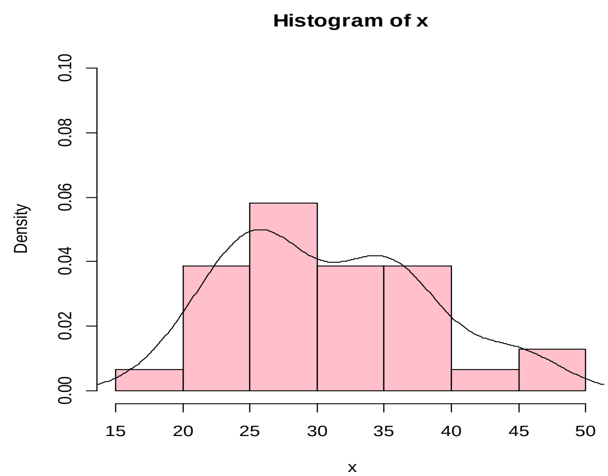

Figure 5: histogram for data 2

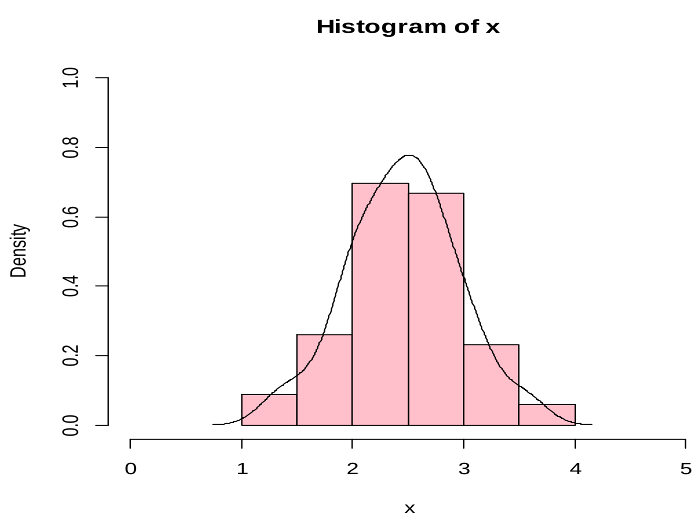

Figure 6: histogram for data 3

Table 7:
Summary statistics of data sets 1, 2 and 3

Data Set	Estimator	Mean	Variance	Coefficient of Variation	Coefficient of Skewness	Coefficient of Kurtosis
Data 1	1.46899	1.805781	1.961309	0.7755466	0.002243082	9.955338
Data 2	0.1283209	30.81112	243.1255	0.5060667	1.055789	4.556607
Data 3	1.194522	2.546862	3.051602	0.6858965	0.01412764	9.028354

From Tables 4-6, it can be observed that the SWG shows lower values of the criterion as compared to the competitive distributions such as Pranav, Ishita, Sujatha, Akash, Shankar, exponential and Lindley distributions. These said distributions are developed on the two-component mixture or three-component mixture of exponential and gamma distributions, but the proposed model SWG is more flexible as compared to these models.

From Table 7, it is observed that data-1 is almost symmetric, leptokurtic, and has less variation. The average survival time for guinea pigs is 1.81 days. Data-2 is positively skewed, leptokurtic and has more variation. The average lifetime of a glass in the aircraft is 30.81. Data-3 is slightly positively skewed, leptokurtic and has less variation in it. The average gauge length of carbon fibre is 2.55 mm.

Figure 4 for data 1, Figure 5 for data 2 and Figure 6 for data 3 shows the visual fitting of SWG distribution on the three under study data sets and which comply with the results presented in Tables 4-6.

8. Comparison

The formulation of the proposed Sum of Weighted Gamma (SWG) distribution draws conceptual motivation from previously developed lifetime models such as Lindley, Shankar, Pranav, Akash, and Sujatha distributions. These existing models primarily rely on two-component or three-component mixture structures involving the exponential and gamma distributions. In the two-component mixtures, a single fixed value for the gamma shape parameter (typically 2 or 3) is used alongside an unknown scale parameter. In three-component mixtures, two fixed shape parameters are adopted with a common unknown scale parameter. In contrast, the proposed SWG distribution extends this framework by incorporating four fixed values of the gamma shape parameter, combined with an unknown scale parameter. Each gamma component is assigned a distinct weight, providing greater modelling flexibility and enabling the distribution to capture a wider variety of data patterns. This multi-weighted structure significantly enhances the adaptability of the model compared to existing two- and three-component counterparts.

A comparative performance assessment of the SWG distribution with other well-known single-parameter models, including the Pranav, Ishita, Sujatha, Akash, Shankar, exponential, and Lindley distributions, is conducted using three real lifetime datasets. As shown in Tables 3–5, the SWG distribution consistently exhibits lower values of selection criteria (e.g., AIC, BIC), indicating a superior fit across all datasets.

A summary of the data characteristics, as presented in Table 6, further underscores the versatility of the SWG model:

- Data-1 (guinea pig survival time): Exhibits an almost symmetric, leptokurtic distribution with low variation. The average survival time is 1.81 days.
- Data-2 (aircraft glass lifetime): Shows a positively skewed, leptokurtic distribution with high variation. The average lifetime is 30.81 units.
- Data-3 (carbon fibre gauge length): Displays a slightly positively skewed, leptokurtic pattern with low variation. The average Gauge length is 2.55 mm.

These datasets represent varying distributional shapes and dispersion levels, and yet the SWG distribution demonstrates superior adaptability in modelling all three. The results confirm that, due to its richer parameter structure and flexible weighting scheme, the SWG distribution outperforms other single-parameter models, especially those based solely on limited gamma-exponential mixtures.

9. Conclusion

In this research article, we developed a generator to develop a class of continuous distributions. We used this generator to develop a new distribution named the sum of weighted gamma (SWG) distribution. We selected some weights and gamma pdfs with different combinations of their parameters in the generator to develop SWG distribution. The proposed SWG distribution is obtained by weighing the gamma distribution with varying shape parameters from 1 to 4 and fixing the scale parameter as $'\theta'$. Various mathematical properties of the SWG distribution, like moments, skewness, kurtosis, incomplete moments, mean deviations, Shannon entropy, order statistics, upper record values, reliability, and inequality measures (Lorenz and Bonferroni curves, Gini and Bonferroni indices) have been derived. The parameter of the proposed density is estimated by MLE, and a simulation study is conducted to check the performance of its estimated parameter. Finally, we modelled SWG distribution on three real lifetime data sets. The AIC, AICC and BIC criteria are presented to compare the applicability of the proposed density over some well-known existing single-parameter distributions. It is observed that the SWG distribution is more applicable and efficient as compared to models like Parnav, Akash, Ishita, Sujatha, Shankar, Lindley, and the exponential distribution. It is concluded that the proposed model is significant and more flexible on lifetime data sets as compared to the aforesaid models. Moreover, it is recommended that the proposed summing of weights method can be used to find various new distributions by changing the mixing proportions of weights.

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Declaration

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