SCOPUA Journal of
Applied Statistical Research
Vol. 1, Issue 2, August 2025
DOI: https://doi.org/10.64060/JASR.v1.i2.2

Power Comparison of Modality Tests

Muhammad Imran Arshad[1], Sadaf Khan [2*], Farrukh Jamal [2], Shahina Aslam[3]

[1]Department of Statistics Govt. S. E. College Bahawalpur
[2]Department of Statistics, The Islamia University of Bahawalpur, Bahawalpur
[3]Department of Economics Govt. S. E. College Bahawalpur
* Corresponding Email: smkhan6022@gmail.com

Received: 12 March 2025 / Revised: 11 July 2025 / Accepted: 13 July 2025 / Published online: 16 August 2025

ABSTRACT

In this paper, the power of each test of unimodality/multimodality is estimated. The power of each test is estimated on the basis of the alternative hypothesis that there is bimodality. The Power Curve and Power Envelope of each test of unimodality /multimodality are also shown by using a graphical representation. The most stringent test of all the four tests of unimodality/multimodality is also recognised and finally finds the Conclusion about the most powerful, best test, worst test and most stringent test among these four stated tests.

Keywords:Modality Tests; Size of the test; Silverman Bandwidth Test; Proportional Mass (PM) Test; Excess Mass (EM) Test; Monte Carlo Simulation Technique

1. Introduction

The study of unimodality and multimodality in statistical distributions is fundamental in various fields, including economics, biology, and social sciences. Modality tests, such as the Hartigan Dip Test, Silverman Bandwidth Test, Proportional Mass (PM) Test, and Excess Mass (EM) Test, are widely used to determine the number of modes in a dataset. Understanding the power of these tests under different conditions is crucial for selecting the most appropriate test for empirical applications.

Modality tests are essential tools for identifying the underlying structure of data distributions. The Hartigan Dip Test (Hartigan & Hartigan, 1985) and the Silverman Bandwidth Test (Silverman, 1986) are among the most commonly used tests. Recent studies, such as Henderson et al. (2006), have extended these tests to weighted and calibrated modes, enhancing their applicability.

The PM Test and EM Test (Mueller & Sawitzki, 1991) offer alternative approaches, particularly useful for large datasets. However, their performance varies with sample size and distribution parameters, as highlighted by Cheng and Hall (1998). Bianchi (1997) applied nonparametric multimodality tests to economic convergence studies, while Hall and Ooi (2004) introduced probabilistic approaches to density shape analysis.

Recent advances in modality testing have addressed critical limitations of classical methods. Extending the work of Fisher & Marron (2001), Ameijeiras-Alonso et al. (2019) *p*roposed a calibrated excess mass test for irregular densities, improving the detection of non-Gaussian multimodality. **Jones and Pewsey (2020) generalised distributions for unimodality testing, while Loader (2021) optimised bandwidth selection for multimodality detection in kernel-based methods.** Dümbgen & Walther (2020) proposed a new critical bandwidth selection for Silverman-type tests, *r*esolving Loader's (2021) bandwidth dependence issue. Chen & Zhang (2021) introduced multimodality tests for high-dimensional data. Meyer (2022) offered finite-sample corrections for Hartigan's Dip Test. Pomenti et al. (2023) initiated robust modality tests for contaminated data, overcoming the proposed test's sensitivity to outliers by Bianchi (1997).

While foundational work established key tests (Dip and Bandwidth tests), several unresolved challenges persist. For instance, Silverman (1986) noted that kernel-based tests (e.g., Silverman Test) may fail to detect subtle multimodality in small samples. Bianchi (1997) focused on economic data, leaving non-normal applications unexplored. Loader (2021) highlighted that existing tests rely on ad hoc bandwidth choices, which can inflate Type-I or Type-II errors. Mueller & Sawitzki (1991) found that Excess Mass (EM) tests degrade with increasing variance. Jones & Pewsey (2020) generalised unimodality tests but did not evaluate weighted mixtures ($\alpha \neq 0.5$). Zaman (1996) proposed stringency metrics but lacked empirical validation, among others.

The motivation for this study stems from the need to identify the most robust and powerful modality test for detecting bimodality, especially in small and large samples, by comparing the power of the tests for varying sample sizes ($\mathrm{n} = 50 – 200$). Further, our study resolves the issue of ad hoc bandwidth selection by fixing bandwidths to isolate test performance, for systematically varying σ^2. Additionally, the work under consideration evaluates the weighted mixtures as our data generating process (DGP) includes $\alpha \in [0.25, 0.65]$, filling the existing gap in the work of Jones & Pewsey (2020). Moreover, we operationalise the stringency metrics in Tables 4–6 by providing the empirical validation initially proposed by Zaman (1996) but without pragmatic support. For brevity, the significance of the work under consideration lies in providing researchers with clear guidelines for test selection, ensuring accurate results in empirical analyses. The paper is organised as follows: Section 2 provides the background and literature review. Section 3 describes the data-generating process. Sections 4 and 5 present the power comparison and power envelope analysis, respectively. Section 6 discusses the most stringent test, and Section 7 concludes the paper.

2. The DGP for the power of each test

The DGP of Hartigan Dip Test, Silverman Bandwidth test, Proportional Mass test and Excess Mass test have been taken from the mixture of two normal densities. The density function of the mixture of two normal distributions for making bimodality is explained as:

$X \sim N(\mu_1, \sigma_1^2)$ *where* X follows the unimodal normal distribution with parameters μ_1 and σ_1^2.
$Y \sim N(\mu_2, \sigma_2^2)$ *where* Y follows the unimodal normal distribution with parameters μ_2 and σ_2^2

$$Z = \begin{Bmatrix} \text{X with probablity } \alpha \\ \text{Y with probability 1-}\alpha \end{Bmatrix} \qquad 0 \le \alpha \le 1$$

Where "Z" is the bimodal distribution and "α" uses as the probability weight of each unimodal density and its value lies between 0 and 1.

2.1 Comparison of Power of Each Test

There are five parameters for making bimodality with the mixture of two normal distributions. These parameters are $\mu1$, $\mu2$, $\sigma1$, $\sigma2$, α. The Powers of each test of unimodality/multimodality are calculated of all of the 81 combinations of alternative. The powers computation of the Hartigan Dip Test, the Silverman Bandwidth test, the Proportional Mass test and the Excess Mass test are obtained from a Monte Carlo sample size of 10,000. The calculations of the powers of four tests of unimodality/ multimodality are shown in the following Tables.

Table 1:

Comparison of powers for a sample size 50 and only μ_2 is varying

Mixture of two Normal densities
Power of Tests for Sample Size 50

μ_2	Hartigan Dip Test	Silverman Test	PM Test	EM Test	MPA
1	5.05	11.3	4	5.8	11.3
2	9.14	44	11	9.2	44
3	30.09	97.4	10	29.8	97.4
4	76.2	100	12	73.8	100
5	96.92	100	19	97.9	100
6	99.73	100	42	99.9	100
7	99.98	100	53	100	100
8	99.99	100	53	100	100
9	100	100	49	100	100

Table 1 explains that when the location parameter μ2of the density increases and all other parameters $(\mu 1, \sigma 1, \sigma 2, \alpha)$ are kept constant the value of the power also increases rapidly in the Hartigan test, Silverman Test and Excess Mass Test at a sample size of 50. But PM Test provides low power which is up to 4% to 49%. Thus, it is concluded that the Silverman Test is the most powerful compared to the other.

Table 2 shows that when the location parameter μ2 of the density increases and all other parameters $(\mu 1, \sigma 1, \sigma 2, \alpha)$ are kept constant the value of the power of the Hartigan test, the Silverman Test and Excess Mass Test increases step by step as the sample size 100. But PM Test provides low power as compared to others (i.e.20% to 63%). Hence, it is also established that the Silverman Test is the most powerful compared to the other.

The empirical findings in Table 3 indicate that as the location parameter μ_2 as the bimodal density increases and all other parameters (μ1, σ1, σ2, α) are kept constant, the value of the power of the Hartigan test, Silverman Test and Excess Mass Test increases slowly at sample size 200. But the power of the PM test shoots up to above 90%, because of the large sample size. In this situation, the PM test looks the most powerful test compared to the other.

Table 2:

Comparison of powers for a sample size is 100 and only μ_2 is varying

Mixture of two Normal densities

Power of Tests for Sample Size 100					
μ_2	Hartigan Dip Test	Silverman Dip Test	PM Test	EM Test	MPA
1	5.39	7	20	4.1	20
2	5.4	15	8	4.4	15
3	16.62	24	17	16.7	24
4	83.31	100	16	83.1	100
5	99.9	100	10	99.9	100
6	100	100	16	100	100
7	100	100	41	100	100
8	100	100	46	100	100
9	100	100	63	100	100

Table 3:

Comparison of powers for sample size is 200 and only μ_2 is varying

Mixture of two Normal densities					
Power of test for Sample Size 200					
μ2	Hartigan Dip Test	Silverman Test	PM Test	EM Test	MPA
1	5.12	12	98	6.1	98
2	5.02	33	94	3.9	94
3	20.78	48	95	21.2	95
4	97.5	100	97	97.5	100
5	100	100	98	100	100
6	100	100	91	100	100
7	100	100	95	100	100
8	100	100	100	100	100
9	100	100	100	100	100

Table 4 indicates that when the location parameter μ2 increases, the density makes bimodality and when the scale parameter σ2 increases, the density starts to lose the shape of bimodality and move to re-emerge to unimodality. All other parameters (μ1, σ1, α) are kept constant at a sample size of 50. The values of the power of the Hartigan Dip test are 6% to 26% and the Excess Mass Test is decreasing from 4% to 27%. Because the Hartigan dip Test and Excess Mass Test do not pick up small bumps efficiently in a small sample. And the values of the power PM Test provide 7% to 58% due to a small sample. But the Silverman Test looks the most powerful test compared to the other, because the Silverman test catches the small bumps very efficiently, even in a small sample, which is why the values of the power of the Silverman test perform well.

Table 4:

Comparison of powers for a sample size is 50 when μ_2 and σ^2 are varying

Mixture of two Normal densities						
Power for Sample Size 50						
μ_2	σ2	Hartigan Dip Test	Silverman Test	PM Test	EM Test	MPA
1	1	6.04	3	7	4.8	7
2	1.5	5.44	35	11	5.7	35
3	2	6.83	78	14	8.4	78
4	2.5	9.06	88	24	10.4	88
5	3	14.03	90	31	13	90
6	3.5	17.61	91	35	15.3	91
7	4	20.29	96	44	22.3	96
8	4.5	23.57	100	56	26.6	100
9	5	26.5	100	58	26.7	100

According to the findings reported in Table 5, it is reported that when the location parameter μ2 increases, the density makes bimodality and when the scale parameter $\sigma 2$ increases, the density starts to lose the shape of bimodality and move to re-emerge to unimodality. All other parameters (μ1, σ1, α) are kept constant at a sample size of 100. The values of the power of the Hartigan Dip test and Excess Mass Test are increasing from 6% to 100% and 5% to 100% respectively, due to an increase in the sample size. But the PM Test goes down very low due to a small sample. And the power of the Silverman Test increases 15% to100% because this test catches the very small bumps very efficiently, even in a small sample. That is why it is the most powerful test.

Table 6 specifies that when the location parameter μ2 increases, the density makes bimodality and when the scale parameter σ2increases, the density starts to lose the shape of bimodality and move to re-emerge to unimodality. All other parameters (μ1, σ1, α) are kept constant at sample size 200, and all other parameters (μ1, σ1, α) are kept constant. As the sample size increases, the values of the power of the Hartigan Dip test and Excess Mass Test are 6% to 60% due to the decrease in the bumps of the density. And the values of the power PM Test are increasing from 4% to 50% due to a large sample, but with very small bumps. But again Silverman Test looks the most powerful test compared to the other, because the Silverman test catches the small bumps very efficiently, even in small samples as well as large samples.

Table 5:

Comparison of powers for a sample size is 100 when μ_2 and σ^2 are varying

Mixture of two Normal densities						
Power of test for Sample Size 100						
μ_2	σ^2	**Hartigan Dip Test**	**Silverman Test**	**PM Test**	**EM Test**	**MPA**
1	1	5.69	15.6	9.3	5.1	15.6
2	1.5	11.33	57.4	1.7	8.9	57.4
3	2	67.65	98.9	1	64.5	98.9
4	2.5	99.86	100	0	99.9	100
5	3	100	100	0	100	100
6	3.5	100	100	0.1	100	100
7	4	100	100	0.3	100	100
8	4.5	100	100	0.6	100	100
9	5	100	100	0.7	100	100

Table 6:

Comparison of powers for sample size is 200 when μ2 and σ2 are varying

Mixture of two Normal densities						
Power of test for Sample Size 200						
μ2	**σ2**	**Hartigan Dip Test**	**Silverman Test**	**PM Test**	**EM Test**	**MPA**
1	1	5.57	17	10.1	6.2	17
2	1.5	4.02	67	3.79	4	67
3	2	4.63	91	9.76	5.8	91
4	2.5	10.39	100	4.72	11.4	100
5	3	18.96	100	2.73	18.3	100
6	3.5	29.24	100	9.67	29.4	100
7	4	40.44	100	26.99	39.4	100
8	4.5	48.16	100	45.46	50	100
9	5	55.98	100	52.54	50.9	100

Table 7 designates that as the μ2 and α increase, the values of the power of Hartigan Dip test, Excess Mass Test and Silverman are also increases 5% to 100%, 6% to 100% and 3% to 100% respectively and all other parameters (μ1, σ1, and σ2) are kept constant at sample size 50. Where α is the probability weight of the bimodal density, which means how much of the data goes to the two respective unimodal distributions to make bimodality. The values of the power of the PM Test decrease due to a small sample. However, the power of Silverman is also looking good compared to the other tests.

Table 7:
Comparison of powers for a sample size is 50 when μ2 and α are varying

Mixture of two Normal densities						
Power of test for Sample Size 50						
μ2	**α**	**Hartigan Test**	**Silverman Test**	**PM Test**	**EM Test**	**MPA**
1	0.25	4.75	3	3	6.5	6.5
2	0.3	5.74	11	2	5.5	11
3	0.35	14.72	21	5	15.1	21
4	0.4	61.56	67	13	65	67
5	0.45	95.4	89	33	95.6	95.6
6	0.5	99.8	95	38	99.7	99.8
7	0.55	99.9	97	52	99.7	99.9
8	0.6	99.15	99	46	100	100
9	0.65	99.94	100	27	99.7	100

Table 8:
Comparison of powers for a sample size is 100 when μ2 and α are varying

Mixture of two Normal densities						
Power of test for Sample Size 100						
μ2	**α**	**Hartigan Test**	**Silverman Test**	**PM Test**	**EM Test**	**MPA**
1	0.25	5.12	7	21	4.1	21
2	0.3	5.02	15	4	4.4	15
3	0.35	20.78	24	5	16.7	24
4	0.4	97.5	100	6	83.1	100
5	0.45	100	100	13	99.9	100
6	0.5	100	100	22	100	100
7	0.55	100	100	34	100	100
8	0.6	100	100	42	100	100
9	0.65	100	100	13	100	100

It is conveyed in Table 8 that as the μ2 and α increase, the values of the power of Hartigan Dip test, Excess Mass Test and Silverman are also increases 5% to 100%, 4% to 100% and 7% to 100% respectively at sample size 100 and all other parameters (μ1, σ1, and σ2) are kept constant. Where α is the probability weight of the bimodal density, which means how much of the data goes to the two respective unimodal distributions to make the bimodality. And the values of the power PM Test decrease due to a small sample. Here again, the power of Silverman is also looking good.

Table 9:
Comparison of powers for sample size is 200 when μ2 and α are varying

Mixture of two Normal densities						
Power of test for Sample Size 200						
μ2	**α**	**Hartigan Test**	**Silverman Test**	**PM Test**	**EM Test**	**MPA**
1	0.25	5.39	12	99.6667	6.1	99.6667
2	0.3	5.4	33	94	3.9	94
3	0.35	16.62	48	87	21.2	87
4	0.4	83.31	100	94	97.5	100
5	0.45	99.9	100	97.3333	100	100
6	0.5	100	100	97	100	100
7	0.55	100	100	98	100	100
8	0.6	100	100	99.3333	100	100
9	0.65	100	100	99.3333	100	100

Finally, Table 9 specifies that as the μ2 and α increase, the values of the power of Hartigan Dip test, Excess Mass Test and Silverman are also increases 5% to 100%, 6% to 100% and 12% to 100% respectively at sample size 200 and all other parameters (μ1, σ1, and σ2) are kept constant. Where α is the probability weight of the bimodal density, which means how much of the data goes to the two respective unimodal distributions to make the bimodality. But the values of the power PM Test increase almost at all the alternatives above 90% to 100% due to the large sample. It is concluded that the PM Test is performing well a large sample. So here PM test is the most powerful test.

Overall, it is concluded that almost all the alternatives and sample sizes Silverman Bandwidth test is the most powerful test except in two cases when the sample size is large i.e.200, that the Proportional Mass test is the most powerful test. It is further concluded that the Silverman Test is powerful in large samples as well as small samples. But the Proportional Mass test is suitable for large samples only.

3. Approximating the Power Envelope

In this study, all 81 alternatives of the tests are used to determine the maximum power among all four tests at different sample sizes and different alternatives as a Maximum Power Approximation of (MPA). This is called the Approximating the Power Envelop. For explaining the graphical depiction, the research takes the value of departure from unimodality along the X-axis and the powers of each test are along the Y-axis. Also, plot the Maximum Power Approximation (MPA) for power comparison of different sizes and different alternatives are also given below:

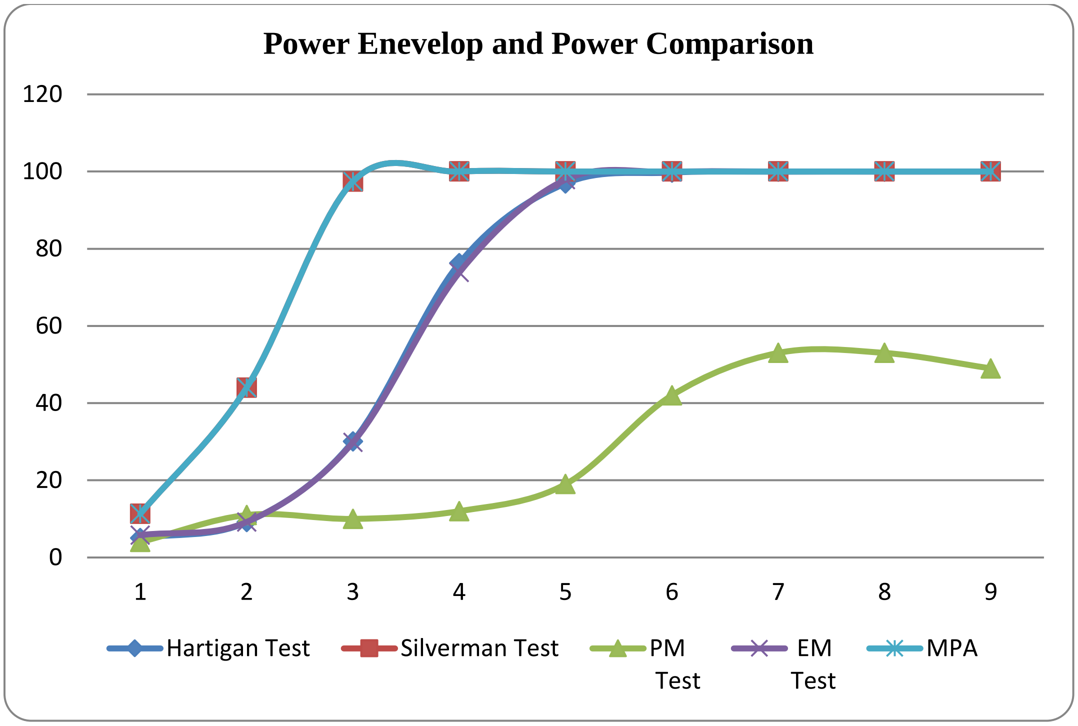

Figure 1: Power Envelop and Power Comparison of each test for n=50 with μ2 varying

Figure 1 plots the departure from unimodality along the X-axis and the value of the power of the test along the Y-axis. The value of the power of the Hartigan test, the Silverman Test and the Excess Mass Test also increases rapidly and the PM Test provides low power at a sample size 50. But the power of the Silverman Test looks the most powerful test compared to the other.

Figure 2 displays the departure from unimodality along the X-axis and the value of the power of the test is along the Y-axis. The value of the power of the Hartigan test, the Silverman Test and the Excess Mass Test also increases rapidly and the PM Test provides low power at a a sample size of 100. But the power of the Silverman Test looks the most powerful test compared to the other.

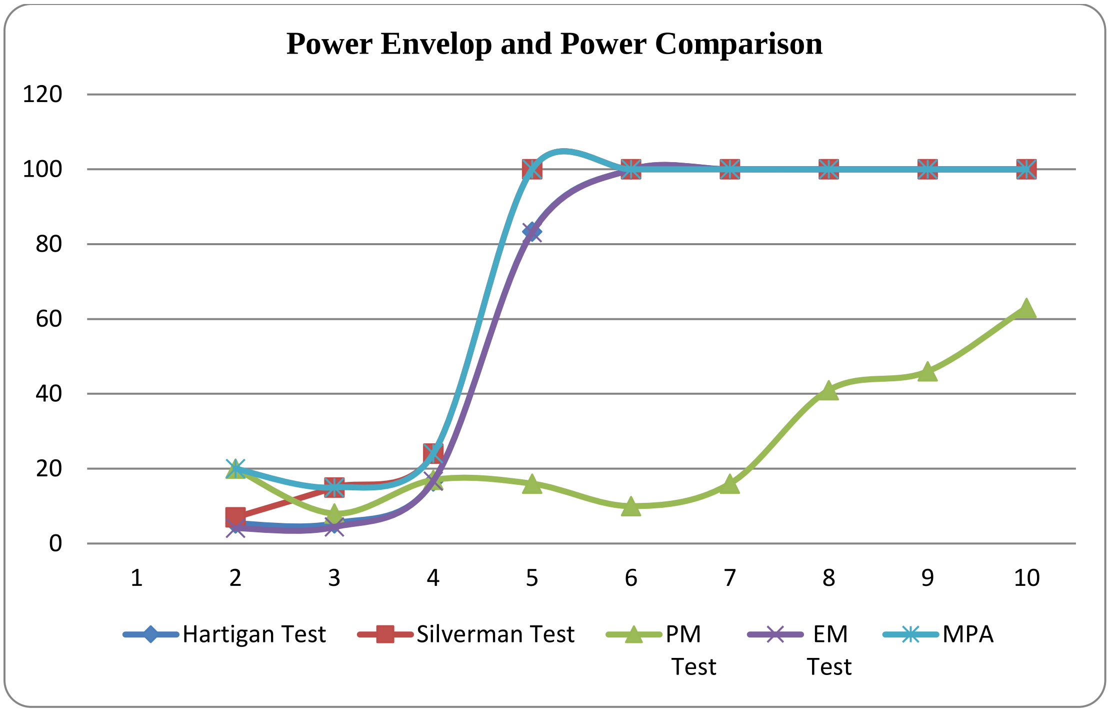

Figure 2: Power Envelop and Power Comparison of each test for n=100 with µ2 varying

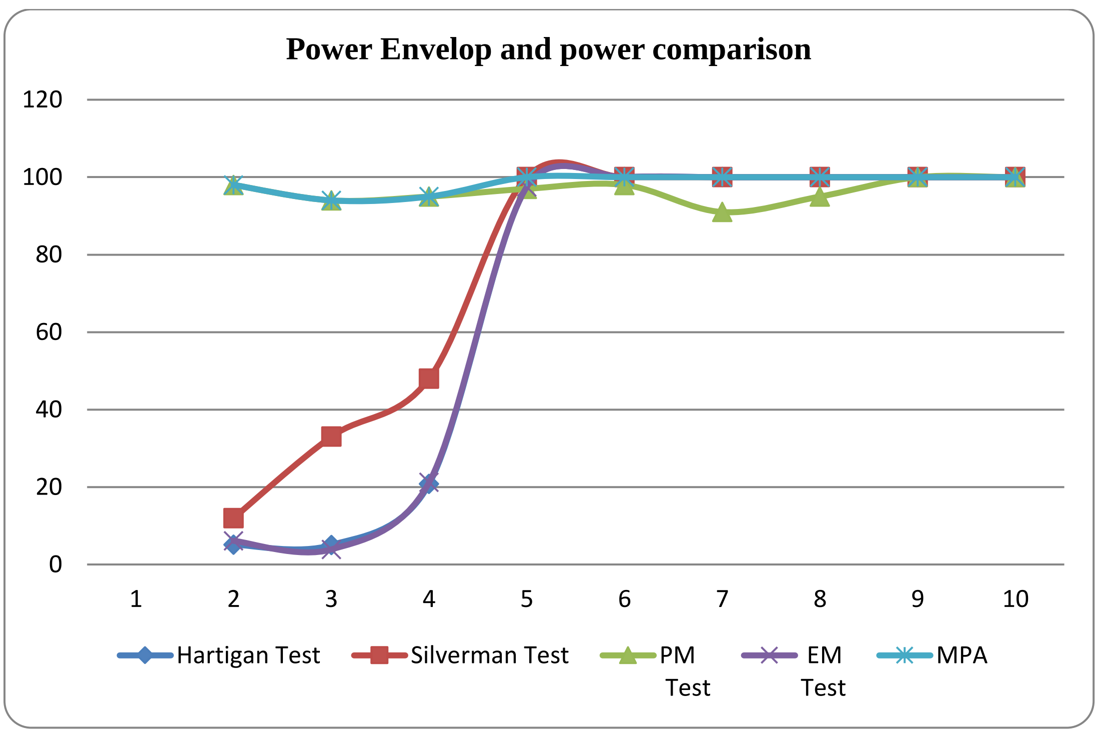

Figure 3: Power Envelop and Power Comparison of each test for n=200 with µ2 varying

Figure 3 depicts the departure from unimodality along the X-axis and the value of the power of the test is along the Y-axis. The value of the power of the Hartigan test, the Silverman

Test and the Excess Mass Test also increases rapidly and the power of the PM Test shoots up at a sample size of 200. So PM Test looks the most powerful test compared to the other because of the large sample.

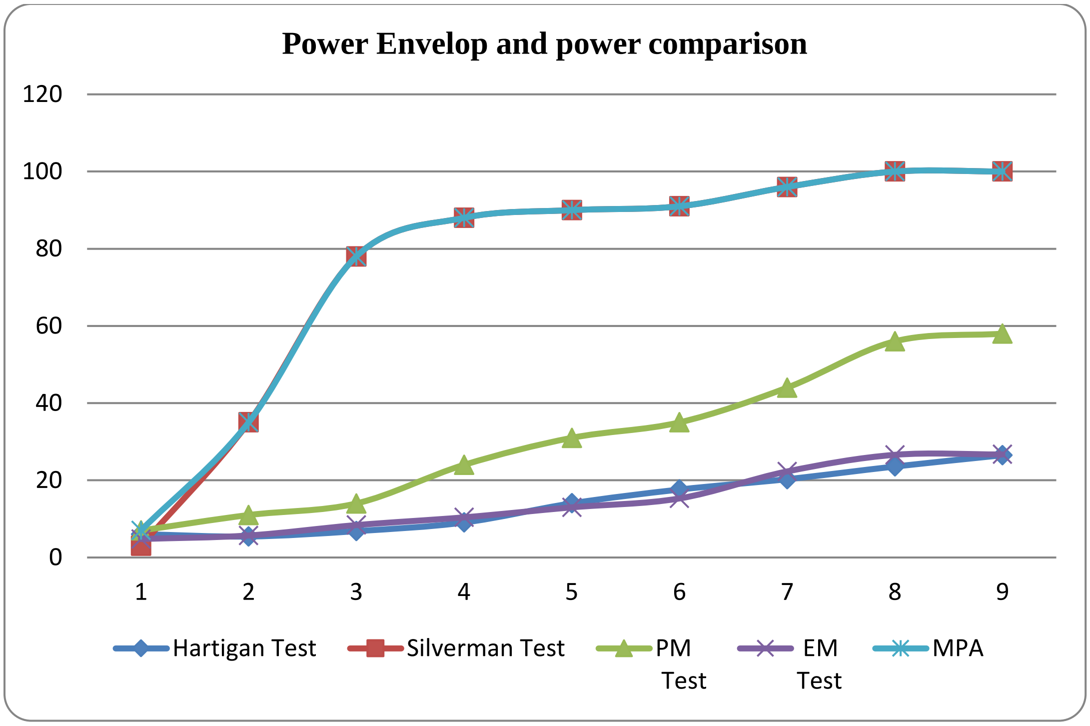

Figure 4: Power Envelop and Power Comparison of each test for n=50 with μ2 and σ2 varying

Figure 4 demonstrates the departure from unimodality along the X-axis and the power of the test along the Y-axis. The value of the power of the Hartigan test and Excess Mass is looking very low as compared to the PM Test at a sample size of 50. Because the Hartigan dip Test and Excess Mass Test do not pick up small bumps efficiently in a small sample. And the PM Test is also not performing very well in small samples and small bumps. But the Silverman Test looks the most powerful test compared to the other, because the Silverman test catches the small bumps very efficiently, even in a small sample, which is why the values of the power of the Silverman test perform well.

The value of departure from unimodality along the X-axis and the value of the power of the test are plotted along the Y-axis. The value of the power of the Hartigan test and Excess Mass increases due to an increase in the sample size and the value of the power of the PM Testis going down. But the PM Test is not performing very well in a small sample as well as small bumps. But the Silverman Test looks the most powerful test compared to the other, because the Silverman test catches the small bumps very efficiently, even in a small sample, which is why the values of the power of the Silverman perform well.

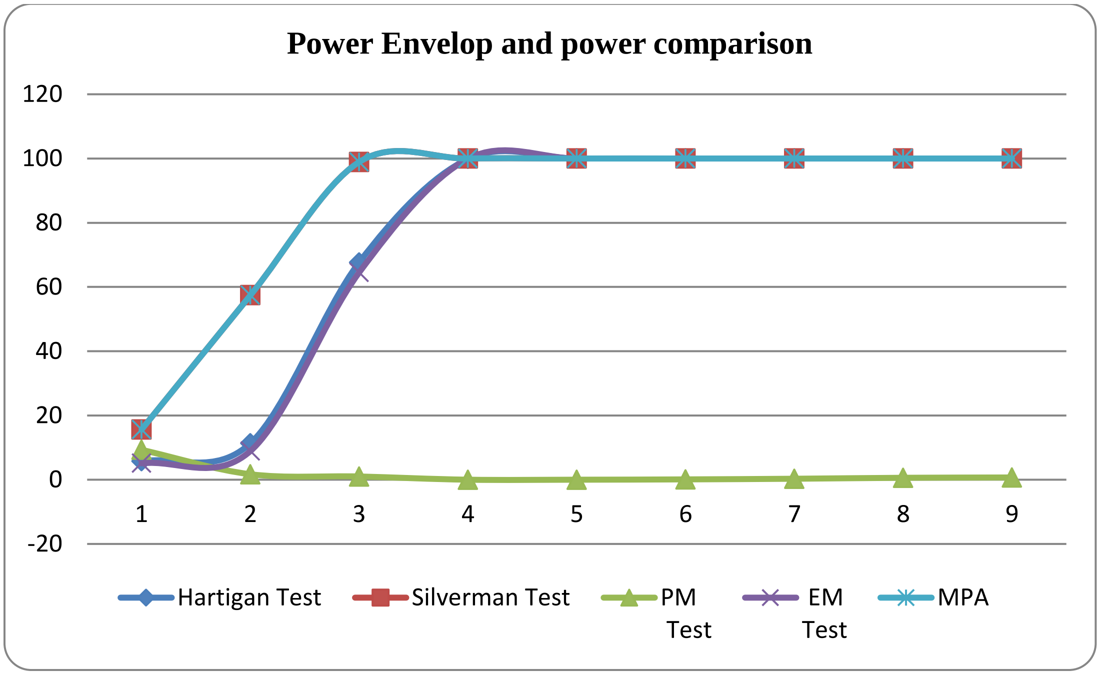

Figure 5: Power Envelop and Power Comparison of each test for n=100 with μ2 and σ2 varying

Figure 5 showcase the value of departure from unimodality along the X-axis and the value of the power of the test are plotted along the Y-axis. The value of the power of the Hartigan test and Excess Mass increases due to an increase in the sample size and the value of the power of the PM Testis going down. But the PM Test is not performing very well in a small sample as well as small bumps. But the Silverman Test looks the most powerful test compared to the other, because the Silverman test catches the small bumps very efficiently, even in a small sample, which is why the values of the power of the Silverman perform well.

The value of departure from unimodality along the X-axis and the value of the power of the test along the Y-axis are plotted in Figure 6. The value of the power of the Hartigan test, Excess Mass and PM Test goes down due to very small bumps at sample size 200. But the Silverman Test looks the most powerful test compared to the other, because the Silverman test catches the small bumps very efficiently, even in large samples, which is why the values of the power of the Silverman perform well.

Figure 7 guides that as the location parameter increases, the values of the power of the Hartigan Dip test, Excess Mass Test and Silverman also increase at sample size 50. And the values of the power of the PM Test decrease due to a small sample. However, the power of Silverman is also looking good as compared to the others.

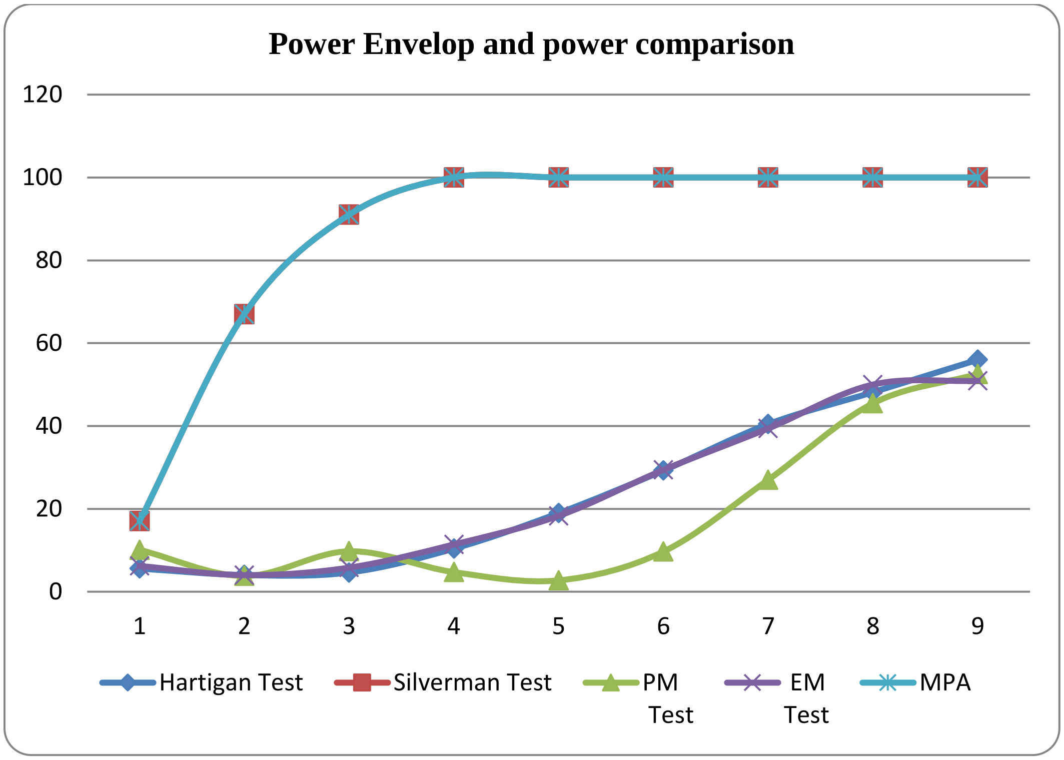

Figure 6: Power Envelop and Power Comparison of each test for n=200 with μ2 and σ2 varying

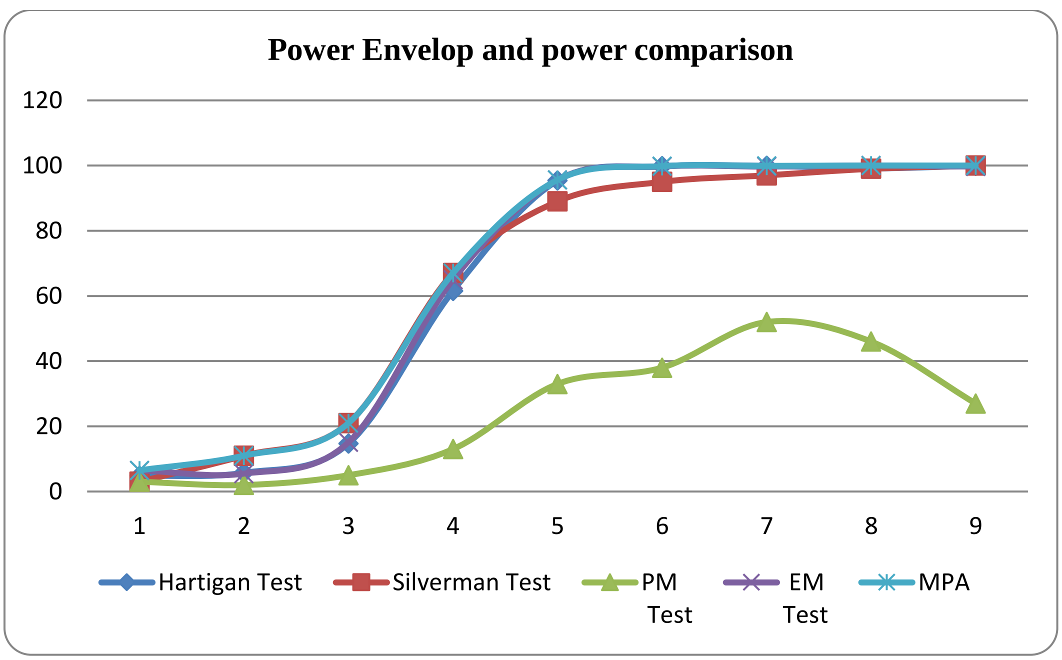

Figure 7: Power Envelop and Power Comparison of each test for n=50 with μ2 and α varying

Figure 8 illustrates that as the location parameter increases, the values of the power of the Hartigan Dip test, Excess Mass Test and Silverman also increase at sample size 100. And the values of the power of the PM Test decrease due to a small sample. However, the power of Silverman is also looking good as compared to the others. Following similar suit, Figure 9 clarifies the values of the power of the Hartigan Dip test, Excess Mass test and Silverman also increase at sample size 200 along with stable results for PM test.

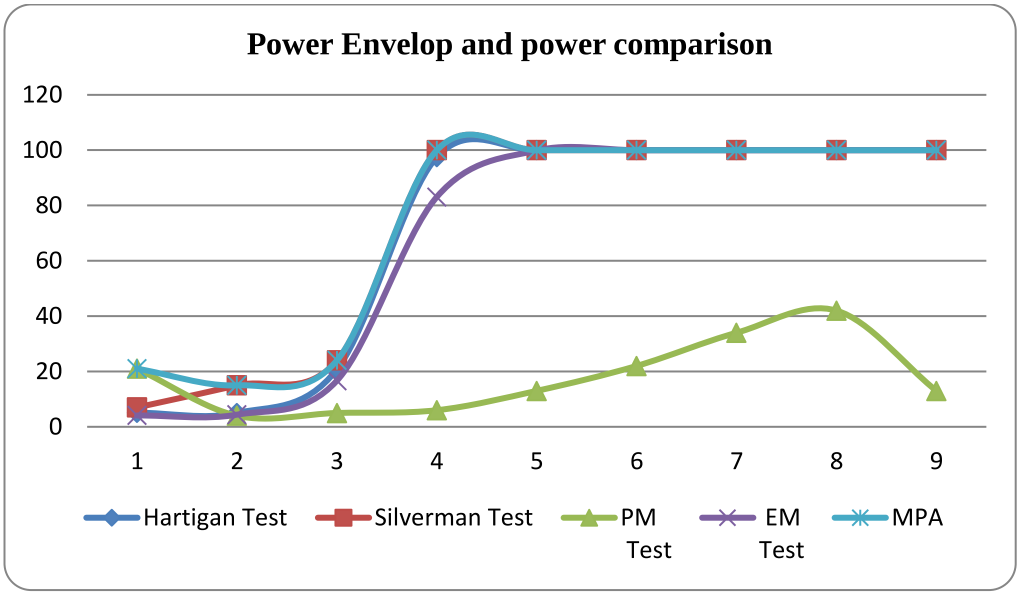

Figure 8: Power Envelop and Power Comparison of each test for n=100 with μ2 and α varying

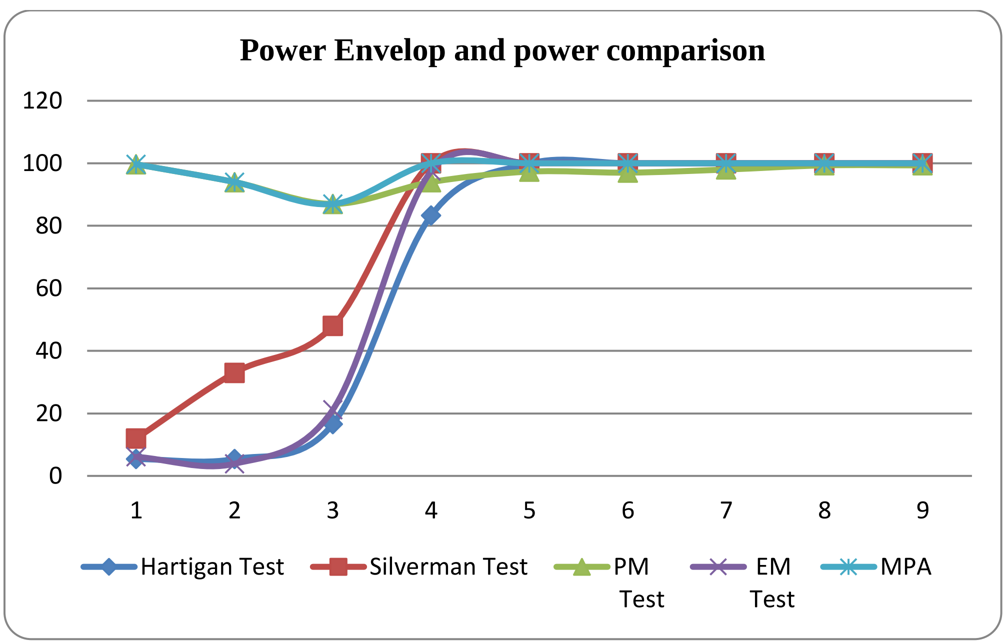

Figure 9: Power Envelop and Power Comparison of each test for n=200 with μ2 and α varying

Overall conclusion of the above graphical comparison showed that the Silverman test is the most powerful test even in large or small sample sizes and large or small bumps. Hartigan and Excess Mass test is also performing well, but not more as Silverman in Small samples and large bumps. While the Proportional Mass test is performing well for large samples and large bumps only. So there are only two cases when the PM test is the most powerful. Finally, it is concluded that Silverman is the best test concerning the comparison of power.

4. Identification of Most Stringent Test

For identification of most stringent test from each test of unimodality/ multimodality, this study uses a method of stringency criteria that is introduced by Dr. Asad Zaman. This method is explained as follows: First, subtract the Power of each test from the Maximum Power Approximation (MPA), then take the value of the maximum of these differences, which is called short shortcomings of the tests Finally get the minimum value of the maximum differences at different alternatives and different sample sizes. A test that has minimum shortcomings is called the most stringent test. In the table below given table **Max** is used for the maximum value of the shortcomings and **MPA** is used for the maximum power approximation. All of these results for different alternatives and different sizes are shown in the following tables:

Table 10:
Comparison of Most Stringent Test for sample size is 50 and only μ2 is varying

	Short Comings			
MPA	**Hartigan DIP Test**	**Silverman Test**	**PMTest**	**EM Test**
11.3	6.25	0	7.3	5.5
44	34.86	0	33	34.8
97.4	67.31	0	87.4	67.6
100	23.8	0	88	26.2
100	3.08	0	81	2.1
100	0.27	0	58	0.1
100	0.02	0	47	0
100	0.01	0	47	0
100	0	0	51	0
Max	67.31	0	88	67.6

Table 10 estimates the shortcomings of the four tests at sample size 50, then finds the maximum values of these shortcomings and finally concludes that the minimum value of that maximum is the most stringent test. So, according to the results presented in the above table, Silverman is the most stringent test because of the minimum shortcomings.

Table 11:
Comparison of the Most Stringent Test for sample size is 100 and only μ2 is varying

Short comings				
MPA	**Hartigan Dip Test**	**Silverman Test**	**PM Test**	**EM Test**
20	14.61	13	0	15.9
15	9.6	0	7	10.6
24	7.38	0	7	7.3
100	16.69	0	84	16.9
100	0.1	0	90	0.1
100	0	0	84	0
100	0	0	59	0
100	0	0	54	0
100	0	0	37	0
Max	16.69	13	90	16.9

Table 11 evaluates the shortcomings of the four tests at sample size 100, then finds the maximum values of these shortcomings and finally concludes that the minimum value of that maximum is the most stringent test. So, according to the results presented in the above table, Silverman is the most stringent test because of the minimum shortcomings.

Table 13:
Comparison of the Most Stringent Test for sample size is 200 and only μ2 is varying

Short comings				
MPA	**Hartigan Dip Test**	**Silverman Test**	**PM**	**EM**
98	92.88	86	0	91.9
94	88.98	61	0	90.1
95	74.22	47	0	73.8
100	2.5	0	3	2.5
100	0	0	2	0
100	0	0	9	0
100	0	0	5	0
100	0	0	0	0
100	0	0	0	0
Max	92.88	86	9	91.9

Table 13 highlights the shortcomings of the four tests at sample size 200, then finds the maximum values of these shortcomings and finally concludes that the minimum value of that maximum is the most stringent test. So, according to the results presented in the above table PM test is the most stringent test because of the minimum shortcomings.

Table 14:
Comparison of the Most Stringent Test for sample size is 50 when μ2 and σ2 are varying

Short comings				
MPA	**Hartigan Dip Test**	**Silverman**	**PM**	**EM**
7	0.96	4	0	2.2
35	29.56	0	24	29.3
78	71.17	0	64	69.6
88	78.94	0	64	77.6
90	75.97	0	59	77
91	73.39	0	56	75.7
96	75.71	0	52	73.7
100	76.43	0	44	73.4
100	73.5	0	42	73.3
Max	78.94	4	64	77.6

Table 14 calculates the shortcomings of the four tests at sample size 50, then finds the maximum values of these shortcomings and finally concludes that the minimum value of that maximum is the most stringent test. So, according to the results presented in the above table Silverman test is the most stringent test because of its minimum shortcomings.

Table 15:
Comparison of the Most Stringent Test for sample size is 100 when μ2 and σ2 are varying

Short comings				
MPA	**Hartigan Dip Test**	**Silverman**	**PM**	**EM**
15.6	9.91	0	6.3	10.5
57.4	46.07	0	55.7	48.5
98.9	31.25	0	97.9	34.4
100	0.14	0	100	0.1
100	0	0	100	0
100	0	0	99.9	0
100	0	0	99.7	0
100	0	0	99.4	0
100	0	0	99.3	0
Max	46.07	0	100	48.5

Table 15 computes the shortcomings of the four tests at a sample size of 100, then finds the maximum values of these shortcomings and finally concludes that the minimum value of that maximum is the most stringent test. So, according to the results presented in the above table Silverman test is the most stringent test because of its minimum shortcomings.

Table 16:
Comparison of the Most Stringent Test for sample size is 200 when μ2 and σ2 are varying

	Short comings			
MPA	**Hartigan Dip Test**	**Silverman**	**PM**	**EM**
17	11.43	0	6.9	10.8
67	62.98	0	63.21	63
91	86.37	0	81.24	85.2
100	89.61	0	95.28	88.6
100	81.04	0	97.27	81.7
100	70.76	0	90.33	70.6
100	59.56	0	73.01	60.6
100	51.84	0	54.54	50
100	44.02	0	47.46	49.1
Max	89.61	0	97.27	88.6

Table 16 determines the shortcomings of the four tests at sample size of 200, then finds the maximum values of these shortcomings and finally concludes that the minimum value of that maximum is the most stringent test. So, according to the results presented in the above table Silverman test is the most stringent test because of its minimum shortcomings.

Table 17:
Comparison of the Most Stringent Test for sample size is 50 when μ2 and α are varying

	Short comings			
MPA	**Hartigan Dip Test**	**Silverman**	**PM**	**EM**
6.5	1.75	3.5	3.5	0
11	5.26	0	9	5.5
21	6.28	0	16	5.9
67	5.44	0	54	2
95.6	0.2	6.6	62.6	0
99.8	0	4.8	61.8	0.1
99.9	0	2.9	47.9	0.2
100	0.85	1	54	0
100	0.06	0	73	0.3
Max	6.28	6.6	73	5.9

Table 17 elucidates the shortcomings of the four tests at sample size of 50, then estimates the maximum values of these shortcomings and finally concludes that the minimum value of that maximum is the most stringent test. In this table, the shortcomings of the Hartigan Dip test, Excess mass Test and Silverman test are looking very close, but according to the results presented in the above table, Excess Mass test is the most stringent test because of the minimum shortcomings. It may have occurred due to a random fluctuation of the density.

Table 18:
Comparison of the Most Stringent Test for sample size is 100 when μ2 and α are varying

Short comings				
MPA	**Hartigan Dip Test**	**Silverman Test**	**PM Test**	**EM Test**
21	15.88	14	0	16.9
15	9.98	0	11	10.6
24	3.22	0	19	7.3
100	2.5	0	94	16.9
100	0	0	87	0.1
100	0	0	78	0
100	0	0	66	0
100	0	0	58	0
100	0	0	87	0
Max	15.88	14	94	16.9

Table 18 highlights the shortcomings of the four tests at a sample size of 100, then estimates the maximum values of these shortcomings and finally concludes that the minimum value of that maximum is the most stringent test. In this table, the shortcomings of the Hartigan Dip test, Excess mass Test and Silverman test are looking very close, but the Silverman test has the minimum short-comings, so it is the most stringent test.

Table 19:
Comparison of the Most Stringent Test for sample size is 200 when μ2 and α are varying

Short comings				
MPA	**Hartigan Dip Test**	**Silverman Test**	**PM Test**	**EM Test**
99.6667	94.2767	87.6667	0	93.5667
94	88.6	61	0	90.1
87	70.38	39	0	65.8
100	16.69	0	6	2.5
100	0.1	0	2.6667	0
100	0	0	3	0
100	0	0	2	0
100	0	0	0.6667	0
100	0	0	0.6667	0
Max	94.2767	87.6667	6	93.5667

Table 19 focuses on the shortcomings of the four tests at a sample size of 200, then estimates the maximum values of these shortcomings and finally concludes that the minimum value of that maximum is the most stringent test. In this table, the shortcomings of the PM test have the minimum shortcomings, so it is the most stringent test.

In general, it is concluded that the Silverman test is the most stringent test as compared to the others, except in the large samples and large bumps only two cases the Proportional mass test is the most stringent.

5. Calculating Power of the Tests

Step-1: Generate a bimodal series with the mixture of two Normal Distributions.

Step-2: Calculate the test statistics of each test and repeat this process 10,000 times.

Step-3: Count the percentage of test statistics greater in magnitude than the critical value.

Step-4: Calculate the power of each test.

Step-5: Repeat these steps from 1 to 4 for different alternatives of the parameters at sample sizes 50, 100 and 200.

The Power of each test of unimodality/multimodality has been estimated based on an alternative hypothesis that isH_1: There is bimodality. The Power Curve and Power Envelope of each test of unimodality/multimodality have also been shown by using a graphical representation. The most stringent test among the four tests of unimodality/multimodality has been investigated through stringency criteria. And finally conclude the most powerful, best test, worst test and most stringent test among these four stated tests.

5.1. Data Generating Process (DGP) for Power of Modality Tests

The DGP of Hartigan Dip Test, Silverman test, Proportional Mass test and Excess Mass test have been taken from the mixture of two normal densities. The density function of the mixture of two normal distributions for making bimodality is explained as:

$X \sim N(\mu_1, \sigma_1^2)$ *where* X follows the unimodal normal distribution with parameters μ_1 and σ_1^2.

$Y \sim N(\mu_2, \sigma_2^2)$ *where* Y follows the unimodal normal distribution with parameters μ_2 and σ_2^2

$$Z = \begin{Bmatrix} X \text{ with probability } \alpha \\ Y \text{ with probability } 1\text{-}\alpha \end{Bmatrix} 0 \leq \alpha \leq 1$$

Where "Z" is a bimodal density function and"α" is the probability weight of the bimodal density, which means how much of the data goes to the two respective unimodal distributions to make bimodality.

5.2. Power Comparison of Modality Tests

There are five parameters for making bimodality with the mixture of two normal distributions. These parameters are μ_1,μ_2,σ_1^2,σ_2^2 and α. The Powers of each test of unimodality/multimodality are calculated for all of the 81 combinations of Alternative. The powersComputation of Hartigan Dip Test, Silverman Bandwidth test, Proportional Mass test and Excess Mass test are obtained from a Monte Carlo sample size of 10,000. The calculations of the powers of four tests of unimodality/ multimodality are shown in the following tables.

Table 20:
Comparison of powers for a sample size of 50 and only μ_2 is varying

Mixture of Two Normal Densities									
Power of Tests for Sample Size 50									
μ_1	σ_1^2	μ_2	σ_2^2	α	**Hartigan Dip Test**	**Silverman Test**	**PM Test**	**EM Test**	**MPA**
0	1	1	1	0.5	5.05	11.3	4	5.8	11.3
0	1	2	1	0.5	9.14	44	11	9.2	44
0	1	3	1	0.5	30.09	97.4	10	29.8	97.4
0	1	4	1	0.5	76.2	100	12	73.8	100
0	1	5	1	0.5	96.92	100	19	97.9	100
0	1	6	1	0.5	99.73	100	42	99.9	100
0	1	7	1	0.5	99.98	100	53	100	100
0	1	8	1	0.5	99.99	100	53	100	100
0	1	9	1	0.5	100	100	49	100	100

From Table 20, we can easily observe that the value of the power also increases rapidly in the Hartigan test, the Silverman Test and the Excess Mass Test at sample size 50. But PM Test provides low power which is up to 4% to 49%. It is concluded that the Silverman Test is the most powerful compared to the other.

When the location parameter $\mu_2$of the density increases and all other parametersμ_1,σ_1^2,$\sigma_2^2$and$\alpha$are kept constant, the value of the power of the Hartigan test, Silverman Test and Excess Mass Test increases step by step at sample size 100. But PM Test provides low power as compared to others (i.e.20% to 63%). From this table, it is also concluded that the Silverman Test is the most powerful compared to the other.

Table 21:
Comparison of powers for a sample size is 100 and only μ_2 is varying

Mixture of Two Normal Densities									
Power of Tests for Sample Size 100									
μ_1	σ_1^2	μ_2	σ_2^2	α	**Hartigan Dip Test**	**Silverman Test**	**PM Test**	**EM Test**	**MPA**
0	1	1	1	0.5	5.39	7	20	4.1	20
0	1	2	1	0.5	5.4	15	8	4.4	15
0	1	3	1	0.5	16.62	24	17	16.7	24
0	1	4	1	0.5	83.31	100	16	83.1	100
0	1	5	1	0.5	99.9	100	10	99.9	100
0	1	6	1	0.5	100	100	16	100	100
0	1	7	1	0.5	100	100	41	100	100
0	1	8	1	0.5	100	100	46	100	100
0	1	9	1	0.5	100	100	63	100	100

Table 21 displays that as the separation between the two means ($\mu_2 - \mu_1$) increases, the power of most tests improves. For well-separated mixtures ($\mu_2 - \mu_1 \geq 3$), Silverman's Test and MPA Test are the most reliable. Hartigan's Dip Test and EM Test are moderately effective but require sufficient separation. The PM Test appears ineffective for this task.

Table 22:
Comparison of powers for sample size is 200 and only μ2 is varying

Mixture of Two Normal Densities									
Power of Tests for Sample Size 200									
μ_1	σ_1^2	μ_2	σ_2^2	α	**Hartigan Dip Test**	**Silverman Test**	**PM Test**	**EM Test**	**MPA**
0	1	1	1	0.5	5.12	12	98	6.1	98
0	1	2	1	0.5	5.02	33	94	3.9	94
0	1	3	1	0.5	20.78	48	95	21.2	95
0	1	4	1	0.5	97.5	100	97	97.5	100
0	1	5	1	0.5	100	100	98	100	100
0	1	6	1	0.5	100	100	91	100	100
0	1	7	1	0.5	100	100	95	100	100
0	1	8	1	0.5	100	100	100	100	100
0	1	9	1	0.5	100	100	100	100	100

Table 22 shows that as the location parameter $\mu_2$of the bimodal density increases and all other parameters $\mu_2, \sigma_2^2$and$\alpha$are kept constant, the value of the power of the Hartigan test, Silverman Test and Excess Mass Test increases slowly at sample size 200. But the power of PM test shoots up to above 90%, because of large sample size. In this situation, the PM test looks the most powerful test compared to the other.

Table 23:
Comparison of powers for sample size is 50 when $\mu_2$and σ_2^2 are varying

Mixture of Two Normal Densities									
Power of Tests for Sample Size 50									
$\boldsymbol{\mu_1}$	$\boldsymbol{\sigma_1^2}$	$\boldsymbol{\mu_2}$	$\boldsymbol{\sigma_2^2}$	$\boldsymbol{\alpha}$	**Hartigan Dip Test**	**Silverman Test**	**PM Test**	**EM Test**	**MPA**
0	1	1	1	0.5	6.04	3	7	4.8	7
0	1	2	1.5	0.5	5.44	35	11	5.7	35
0	1	3	2	0.5	6.83	78	14	8.4	78
0	1	4	2.5	0.5	9.06	88	24	10.4	88
0	1	5	3	0.5	14.03	90	31	13	90
0	1	6	3.5	0.5	17.61	91	35	15.3	91
0	1	7	4	0.5	20.29	96	44	22.3	96
0	1	8	4.5	0.5	23.57	100	56	26.6	100
0	1	9	5	0.5	26.5	100	58	26.7	100

Table 23 illustrates that when the scale parameter $\boldsymbol{\sigma_2^2}$ increases the density, starts to lose the shape of bimodality and move to re-emerge to unimodality. All other parameters μ_1,σ_1^2 and α are kept constant at sample size 50. The values of the power of the Hartigan Dip test are 6% to 26% and the Excess Mass Test is decreasing from 4% to 27%. Because the Hartigan Dip Test and Excess Mass Test do not pick up small bumps efficiently in a small sample. The values of the power PM Test provide 7% to 58% due to a small sample. But the Silverman Test looks the most powerful test compared to the other, because the Silverman test catches the small bumps very efficiently, even in a small sample, which is why the values of the power of the Silverman perform well.

It can be easily observed from Table 24 that as the scale parameter $\boldsymbol{\sigma_2^2}$ increases the density, starts to lose the shape of bimodality and moves to re-emerge into unimodality. All other parameters $\boldsymbol{\mu_1}$,$\boldsymbol{\sigma_1^2}$, α are kept constant at sample size 100. The values of the power of the Hartigan Dip test and Excess Mass Test are increasing from 6% to 100% and 5% to 100% respectively, due to an increase in the sample size. But the Proportional Mass (PM) Test goes down very low due to a small sample. And the power of the Silverman Test increases 15% to100% because this test catches the very small bumps very efficiently, even in a small sample. That is why it is the most powerful test.

Table 24:
Comparison of powers for a sample size is 100 when $\mu_2$and σ_2^2 are varying

Mixture of Two Normal Densities									
Power of Tests for Sample Size 100									
μ_1	σ_1^2	μ_2	σ_2^2	α	**Hartigan Dip Test**	**Silverman Test**	**PM Test**	**EM Test**	**MPA**
0	1	1	1	0.5	5.69	15.6	9.3	5.1	15.6
0	1	2	1.5	0.5	11.33	57.4	1.7	8.9	57.4
0	1	3	2	0.5	67.65	98.9	1	64.5	98.9
0	1	4	2.5	0.5	99.86	100	0	99.9	100
0	1	5	3	0.5	100	100	0	100	100
0	1	6	3.5	0.5	100	100	0.1	100	100
0	1	7	4	0.5	100	100	0.3	100	100
0	1	8	4.5	0.5	100	100	0.6	100	100
0	1	9	5	0.5	100	100	0.7	100	100

Table 25:
Comparison of powers for a sample size is 200 when μ_2 and σ_2^2 are varying

Mixture of Two Normal Densities									
Power of Tests for Sample Size 200									
μ_1	σ_1^2	μ_2	σ_2^2	α	**Hartigan Dip Test**	**Silverman Test**	**PM Test**	**EM Test**	**MPA**
0	1	1	1	0.5	5.57	17	10.1	6.2	17
0	1	2	1.5	0.5	4.02	67	3.79	4	67
0	1	3	2	0.5	4.63	91	9.76	5.8	91
0	1	4	2.5	0.5	10.39	100	4.72	11.4	100
0	1	5	3	0.5	18.96	100	2.73	18.3	100
0	1	6	3.5	0.5	29.24	100	9.67	29.4	100
0	1	7	4	0.5	40.44	100	26.99	39.4	100
0	1	8	4.5	0.5	48.16	100	45.46	50	100
0	1	9	5	0.5	55.98	100	52.54	50.9	100

Table 25 reflects the findings that when the parameter σ_2^2increases the density starts to lose the shape of bimodality and move to re-emerge into unimodality. All other parameters μ_1,σ_1^2, α are kept constant at sample size 200. As the sample size increases, the values of the power of the Hartigan Dip test and Excess Mass Test are 6% to 60% due to the decrease in the bumps of the density. And the values of the power Proportional Mass (PM) Test are increasing from 4% to 50% due to a large sample, but very small bumps. But again Silverman Test looks the most powerful test compared to the other, because the Silverman test catches the small bumps very efficiently, even in small samples as well as large samples.

Table 26:
Comparison of powers for a sample size is 50 when μ_2 and α are varying

Mixture of Two Normal Densities									
Power of Tests for Sample Size 50									
μ_1	σ_1^2	μ_2	σ_2^2	α	**Hartigan Dip Test**	**Silverman Test**	**PM Test**	**EM Test**	**MPA**
0	1	1	1	0.25	4.75	3	3	6.5	6.5
0	1	2	1.5	0.3	5.74	11	2	5.5	11
0	1	3	2	0.35	14.72	21	5	15.1	21
0	1	4	2.5	0.4	61.56	67	13	65	67
0	1	5	3	0.45	95.4	89	33	95.6	95.6
0	1	6	3.5	0.5	99.8	95	38	99.7	99.8
0	1	7	4	0.55	99.9	97	52	99.7	99.9
0	1	8	4.5	0.6	99.15	99	46	100	100
0	1	9	5	0.65	99.94	100	27	99.7	100

Table 26 notes that as the $\boldsymbol{\mu_2}$ and α increase, the values of the power of the Hartigan Dip test, Excess Mass Test and Silverman are also increased 5% to 100%, 6% to 100% and 3% to 100% respectively and all other parameters $\boldsymbol{\mu_1}$,$\boldsymbol{\sigma_1^2}$, and $\boldsymbol{\sigma_2^2}$ are kept constant at a sample size of 50. Where α is the probability weight of the bimodal density, which means how much of the data goes to the two respective unimodal distributions for making the bimodality. And the values of the power of the Proportional Mass (PM) Test decrease due to a small sample. However, the power of Silverman is also looking good compared to the other tests.

Table 27 records that when both the parameters $\boldsymbol{\mu_2}$ and α are increasing, the values of the power of the Hartigan Dip test, Excess Mass Test and Silverman are also increasing 5% to 100%, 4% to 100% and 7% to 100% respectively, at sample size 100 and all other parameters $\boldsymbol{\mu_1}$,$\boldsymbol{\sigma_1^2}$, and $\boldsymbol{\sigma_2^2}$ are kept constant. Where α is the probability weight of the bimodal density, which means how much of the data goes to the two respective unimodal distributions for making bimodality with decreasing values of the PM Test for a small sample. Here again, the power of Silverman is also looking good.

Table 27:
Comparison of powers for a sample size is 100 when $\boldsymbol{\mu_2}$ and α are varying

Mixture of Two Normal Densities									
Power of Tests for Sample Size 100									
μ_1	σ_1^2	μ_2	σ_2^2	α	**Hartigan Dip Test**	**Silverman Test**	**PM Test**	**EM Test**	**MPA**
0	1	1	1	0.25	5.12	7	21	4.1	21
0	1	2	1.5	0.3	5.02	15	4	4.4	15
0	1	3	2	0.35	20.78	24	5	16.7	24
0	1	4	2.5	0.4	97.5	100	6	83.1	100
0	1	5	3	0.45	100	100	13	99.9	100
0	1	6	3.5	0.5	100	100	22	100	100
0	1	7	4	0.55	100	100	34	100	100
0	1	8	4.5	0.6	100	100	42	100	100
0	1	9	5	0.65	100	100	13	100	100

Table 28:
Comparison of powers for a sample size is 200 when $\boldsymbol{\mu_2}$ and α are varying

Mixture of Two Normal Densities									
Power of Tests for Sample Size 200									
μ_1	σ_1^2	μ_2	σ_2^2	α	**Hartigan Dip Test**	**Silverman Test**	**PM Test**	**EM Test**	**MPA**
0	1	1	1	0.25	5.39	12	99.6667	6.1	99.6667
0	1	2	1.5	0.3	5.4	33	94	3.9	94
0	1	3	2	0.35	16.62	48	87	21.2	87
0	1	4	2.5	0.4	83.31	100	94	97.5	100
0	1	5	3	0.45	99.9	100	97.3333	100	100
0	1	6	3.5	0.5	100	100	97	100	100
0	1	7	4	0.55	100	100	98	100	100
0	1	8	4.5	0.6	100	100	99.3333	100	100
0	1	9	5	0.65	100	100	99.3333	100	100

Table 28 summarises the values of the power of the Hartigan Dip test, Excess Mass Test and Silverman as the $\boldsymbol{\mu_2}$ and α are increasing at sample size 200 and all other parameters $\boldsymbol{\mu_1}$,$\boldsymbol{\sigma_1^2}$, and $\boldsymbol{\sigma_2^2}$ are kept constant.. But the values of the power Proportional Mass (PM) Test increase almost at all the alternatives above 90% to 100% due to the large sample. It is concluded that the Proportional Mass (PM) Test is performing well on a large sample. So here PM test is the most powerful test.

6. Conclusion

It is overall concluded that almost all the alternatives and sample sizes Silverman Band-width test is the most powerful test, except for two cases when the sample size is large i.e.200, that the Proportional Mass test is the most powerful test. It is further concluded that the Silverman Test is powerful in large samples as well as small samples. But the Proportional Mass test is suitable for large samples only. For future directives, practitioners are encouraged to refine our DGP with outlier scenarios. Additionally, the performance of these tests can be explored in non-normal distributions and high-dimensional data.

Declaration

Conflict of Study: The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper

Author Contribution Statement:. The manuscript was written through contributions of all authors equally, and RMIA, SK, FJ and SA have given approval to the final version of the manuscript.

Funding Statement: This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Availability of Data and Material: Data will be made available by the corresponding author on reasonable request.

Consent to Publish: All authors have agreed to publish this manuscript in the SCOPUA Journal of Applied Statistical Research (JASR) ISSN(e): 3104-4794.

Ethical Approval: Not Applicable

Consent to Participate: Not Applicable

Acknowledgment: Not Applicable

References

1. Ameijeiras-Alonso, J., et al. (2019). *Journal of Computational and Graphical Statistics, 28*(3), 623–636.
2. Bianchi (1997). Testing for convergence: evidence from nonparametric multimodality tests, *Journal of Applied Econometrics* 12(4): 393-409.
3. Chen *et al.* (2001). A modified likelihood ratio test for homogeneity in finite mixture models. J. R. Stat. Soc. Ser. B, 63, 19–29.
4. Cheng and Hall (1998), “Calibrating the Excess Mass and Hartigan Dip Test Tests of Modality,” Journal of the Royal Statistical Society, Series B, 60, 579-590.
5. Chen, Y., & Zhang, X. (2021). *Journal of Multivariate Analysis*, *182*, 104702.
6. Dümbgen, L., & Walther, G. (2020). *Annals of Statistics, 48*(4), 2058–2081.
7. Fisherand Marron (2001), Mode testing via the excess mass estimate. Biometrica, 88, 499–517.
8. Jones, M. C., & Pewsey, A. (2020). A general family of distributions for unimodality testing. *Journal of Statistical Planning and Inference, 205*, 1–15.
9. Hall and York (2001), On the calibration of Silverman’s test for multimodality. Statist. Sinica, 11, 515–536.
10. Hartigan (1985), Algorithm AS217: Computation of the Hartigan Dip Test statistic to test for unimodality. *Appl. Statist.*34, pp. 320–325.
11. Hartigan and Hartigan (1985), theHartigan Dip Test of unimodality. Ann Statist 13 (1)70-84.
12. Henderson*at al* (2006), “Modes, Weighted Modes and Calibrated Modes: Evidence of Clustering Using Modality Tests,” Journal of Applied Econometrics, forthcoming.
13. Hall and York (2001), “On the Calibration of Silverman’s Test for Multimodality,” StatisticaSinica, 11, 515-536.

14. Hall and OOI (2004), "Attributing a Probability to the Shape of a Probability Density" The Annals of Statistics ,Vol. 32, No. 5, Institute of Mathematical Statistics.
15. Loader, C. R. (2021). Bandwidth selection for multimodality testing. *Computational Statistics & Data Analysis*, *157*, 1–12.
16. Mueller and Sawitzki (1991)Excess mass estimates and tests for multimodality. JASA 86, 738 -746
17. Meyer, M. C. (2022). *Computational Statistics & Data Analysis*, *167*, 107360.
18. Pomenti, S., et al. (2023). *Statistical Methods & Applications*, *32*(1), 1–28.
19. Silverman (1983), `Some properties of a test for multimodality based on kernel density estimates' in J. F. C. Kingman and G. E. H. Reuter (eds), Probability, Statistics, and Analysis, Cambridge University Press, Cambridge.
20. Silverman (1986), Density Estimation for Statistics and Data Analysis, Monographs on Statistics and Applied Probability. 26, Chapman and Hall, London.
21. Zaman A. (1996) "Statistical Foundations for Econometrics Techniques."

Author(s) Bio / Authors' Note

Rana Muhammad Imran Arshad:

R.M.I.A *from Department of Statistics Govt. S. E. College Bahawalpur, Pakistan*.

Email: imranarshad.stat@gmail.com

Sadaf Khan:

S.K *is from Department of Statistics, The Islamia University of Bahawalpur, Bahawalpur*. Email: smkhan6022@gmail.com

Farrukh Jamal:

F.J *is from Department of Statistics, The Islamia University of Bahawalpur, Bahawalpur.*

Email: farrukh.jamal@iub.edu.pk

Shahina Aslam:

S.A *is from Department of Economics Govt. S. E. College Bahawalpur.* Email: shahina_imran@yahoo.com