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A comprehensive study of the extended JCA distribution with properties and applications

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ABSTRACT

The JCA distribution, which was first proposed by Jamal et al. (2021), was expanded in this paper. The new model is called the extended JCA distribution, or EJCA. For the researchers, this will be more adaptable and appealing. Explicit expressions for the moment generating function, the analytical form of the density and hazard, order statistics, and mode are among the mathematical properties that are derived. The maximum likelihood approach is used to estimate the model parameters. To evaluate the effectiveness of the estimation, a simulation study with a range of sample sizes is conducted. Two applications to the real-world data set demonstrate the proposed family's adaptability. We combined the classical exponential distribution with the JCA distribution. The new model is extremely flexible and combines the features of both models, including exponential with some probabilities and JCA. For the new distribution, we derive several mathematical properties and applications to actual data sets.

Keywords: JCA distribution; Mixture distribution; Maximum Likelihood Method

1. Introduction

Since their introduction in 1953, the Lehmann Alternative Methods became very popular to add new parameter(s) in a given probability distribution. The Lehmann Alternatives I (LA1) and II (LA2) methods were used to add more parameters to the distributions to make them more flexible. Modified Fréchet distributions with an additional shape parameter were first produced by generalizing the Fréchet distribution using LA1 and PTM. The Exponentiated Fréchet Distribution is a proportional hazard model that was created by [1] using the Lehmann

Alternative II method (LA2), which was first developed by [2]. Pioneers in applying the cdf for these kinds of applications were [3],[4],[5],[6] in the first half of the 20th century. The Extended JCA distribution was created by adding the parameters α, θ, and k to the distribution. Several distributions were presented in the form of exponents. As an extension of the traditional exponential distribution, the exponentiated exponential (EE) distribution was notably proposed by Gupta et al., whereas the Exponentiated Weibull (EW) distribution was first presented by [7] and has since found widespread use in statistics and probability.

One benefit of the generalized versions of the JCA distribution is that it is possible to study the nonlinear equation that governs the quantile function (qf) of the JCA distribution. Certain values of the parameter k, especially integer values, allow for analytical formulations. Since the pdf and hazard rate function (hrf) are so important for fitting data, we concentrate on them even though each of these functions can be covered separately. Other exponentiated distributions include the Exponentiated Fréchet, Exponentiated Gumbel, Exponentiated Weibull, and Exponentiated Gamma. Similar to how the EE distribution generalizes the exponential distribution, these distributions also extend the classical Gamma, Weibull, Gumbel, and Fréchet distributions. Expressions for variance, skewness, kurtosis, characteristic functions, cumulant generating functions, moments, mean deviation about the mean and median, pdf, CDF, SF, and other characteristics are included in the comprehensive mathematical analysis of each distribution. Also covered are the Gini index, Lorenz curve, order statistics, maximum likelihood function, and other relevant subjects.

Additionally, presented is the Extended JCA distribution, which considers three parameters. A thorough examination of this distribution shows that it is very accurate and flexible for statistical modelling due to its shape behaviour and the corresponding pdf, HRF, moments, asymmetry, and kurtosis measures. The Extended JCA distribution's single parameter is estimated using the greatest likelihood estimate method. The goodness of fit of this distribution has been evaluated using two real-world datasets. The JCA distribution's adjustments were very successful, according to comparisons with failure rate distributions like the Lindley, one-parameter Weibull, exponential, and one-parameter linear models. Furthermore, the JCA distribution's three-parameter version performs better than its one-parameter counterpart. Furthermore, the Kumaraswamy-Fréchet distribution within the KW-G family was derived by [8] using the Fréchet distribution as the base. As a member of the T-X generalized family of distributions, [9] presented the Fréchet-Weibull distribution. [10] proposed the exponential transmuted Fréchet distribution, which is a member of the T-transmuted X family. Using the generalized Kumaraswamy technique, [12] expanded the transmuted Marshall-Olkin Fréchet distribution, which was first presented by [13], into a six-parameter Fréchet distribution. Similarly, [14] examined the Beta Generalized Exponentiated Fréchet (BGEF) distribution, a six-parameter Fréchet model.

Researchers have made extensive use of mixture distributions for a variety of purposes, such as survey analysis and estimation. In some cases, the output of a system can be represented by an inverse Weibull mixture model with negative weights, as mentioned in [15]. [16] investigated the hazard rate and graphical representation of the mixture model of two inverse Weibull distributions. Additionally, [17] examined the statistical properties, graphs, and hazard rates of three inverse Weibull distributions in combination.

To improve model fit and produce more precise predictions, we explore the benefits of applying the extended JCA distribution in statistical modelling in this work. By contrasting the extended JCA distribution's goodness-of-fit metrics with those of conventional models, we show how resilient it is in situations where alternative distributions might not fit data well or might produce skewed estimates. To demonstrate the flexibility and efficacy of the extended JCA distribution in capturing intricate data patterns, its special mathematical characteristics such as its moments, entropy measures, cdf, and pdf are carefully investigated.

We also discuss some of its real-world uses, where the extended JCA distribution routinely fits complex datasets better than traditional models. In domains such as risk management, survival research, and reliability analysis, this distribution offers a more realistic depiction of actual data behaviors that are frequently difficult to simulate with conventional distributions. Through these applications, we show how the distribution can be used to solve a variety of statistical problems, highlighting its potential as a potent tool for practitioners and researchers.

The findings of this study provide a useful framework for individuals looking for more flexible models for complex data scenarios, and they make a substantial contribution to the growing body of knowledge on flexible statistical distributions. The extended JCA distribution creates new opportunities for researchers in different fields by offering a fresh method for modelling and analysis. The paper's conclusion emphasizes how crucial it is to create and use flexible distributions to successfully handle the complexity of real-world data, and it promotes more investigation into the possible uses and future expansions of the JCA distribution family.

2. Purpose of the Study

1. The goal is to present the EJCA, a new model that builds upon the original JCA distribution.
2. To obtain explicit expressions for the density and hazard functions, mode, order statistics, moment generating function (mgf), and other important mathematical characteristics of the EJCA distribution.

3. To apply maximum likelihood estimation (MLE) to the estimation of the parameters of the EJCA model. To estimate parameters using a simulation study with different sample sizes to assess the effectiveness of the MLE approach.
4. To use two real-world datasets to illustrate the EJCA distribution's versatility and flexibility.
5. To create a highly adaptable model that combines aspects of both the JCA distribution and the classical exponential distribution.
6. To determine more mathematical characteristics of the novel model and investigate its usefulness using actual data.

3. The Proposed Model

We present the JCA and EJCA distribution in this section. Its pdf and cdf of JCA distribution are given by

$$g(y;\ k) = (1 + y)^{-k-1}\ (1 + (1-k)y)e^{-y(1+y)^{-k}} \qquad (1)$$

and $$G(y;\ k) = 1 - e^{-y(1+y)-k} \qquad (2)$$

Where $k, y > 0$.

The cdf and pdf of the new proposed exponentiated Generalized family, are given as

$$F(y) = e^{-[-\log(G(y;\,k)^{\alpha})]^{\theta}} \qquad (3)$$

$$f(y) = \frac{\alpha\theta g(y;\,k)}{G(y;\,k)}\ [-\log G(y;\ k)^{\alpha}]^{\theta-1} e^{-[-\log G(y;\,k)^{\alpha}]\ \theta} \qquad (4)$$

By inserting (1) and (2) in (3) and (4) we get the cdf and pdf of EJCA as,

$$F(y) = e^{-\left[-\log\left[1-e^{-y(1+y)^{-k}}\right]^{\alpha}\right]^{\theta}} \qquad (5)$$

and

$$f(y) = \frac{\alpha\theta(1+y)^{-k-1}\ [1+(1-k)y]\ e^{-y(1+y)^{-k}}}{1-e^{-y(1+y)^{-k}}}\ \Big[-\log(1 - e)^{-y(1+y)^{-k}}\Big]^{\theta-1}\ e^{-\left[-\alpha\log(1-e^{-y(1+y)^{-k}}\right]^{\theta}} \qquad (6)$$

The sf, hazard rate, cumulative hazard rate and reverse hazard rate are given below, respectively,

$$S(y) = 1 - e^{-\left[-\log\left[1-e^{-y(1+y)^{-k}}\right]^{\alpha}\right]^{\theta}},$$

$$h(y) = \frac{\frac{\alpha\theta(1+y)^{-k-1}\,[1+(1-k)y]}{\left[e^{y(1+k)^{-k}}-1\right]}\, e^{-[-\alpha \log G(y)]^{\theta}}\, [-\alpha\log(1-e) - y(1+y)^{-k}\,]^{\theta-1}}{1-e^{-\left[-\log\left[1-e^{-y(1+y)^{-k}}\right]^{\alpha}\right]^{\theta}}} \tag{7}$$

$$H(y) = -\log 1 - e^{-\left[-\log\left[1-e^{-y(1+y)^{-k}}\right]^{\alpha}\right]^{\theta}}, \text{ and}$$

$$r(y) = \frac{\frac{\alpha\theta(1+y)^{-k-1}\,[1+(1-k)y]}{\left[e^{y(1+k)^{-k}}-1\right]}\, e^{-[-\alpha \log G(y)]^{\theta}}\, [-\alpha\log(1-e) - y(1+y)^{-k}\,]^{\theta-1}}{e^{-\left[-\log\left[1-e^{-y(1+y)^{-k}}\right]^{\alpha}\right]^{\theta}}}$$

4. The shape of the pdf and hrf of EJCA Distribution

The hrf and pdf, cdf and sf of the EJCA distribution have been plotted for a range of selected parameter values in this section. This enables us to investigate the wide variety of forms and actions that the distribution displays in various scenarios. A wide range of density shapes, including increasing, decreasing, decreasing-increasing-decreasing, symmetric, left-skewed, and right-skewed distributions, are depicted in the plots in Figures 1 & 2, illustrating the family's versatility. On the other hand, the HRF shows bathtub, reverse-J, and constant forms. The CDF is a non-decreasing function, whereas the SF is a decreasing function, as shown in Figure 3.

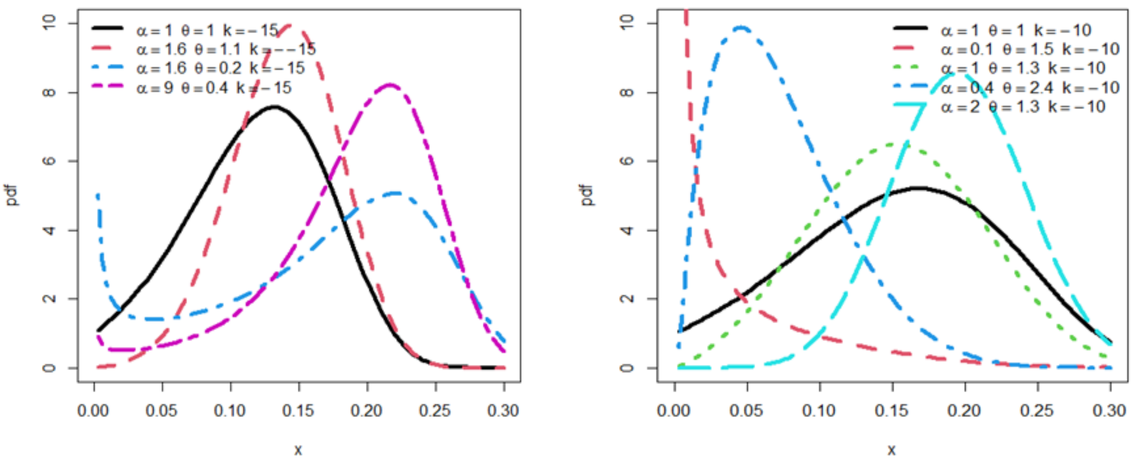

Figure 1: *The graph displays the various forms of the pdf for the EJCA distribution*.

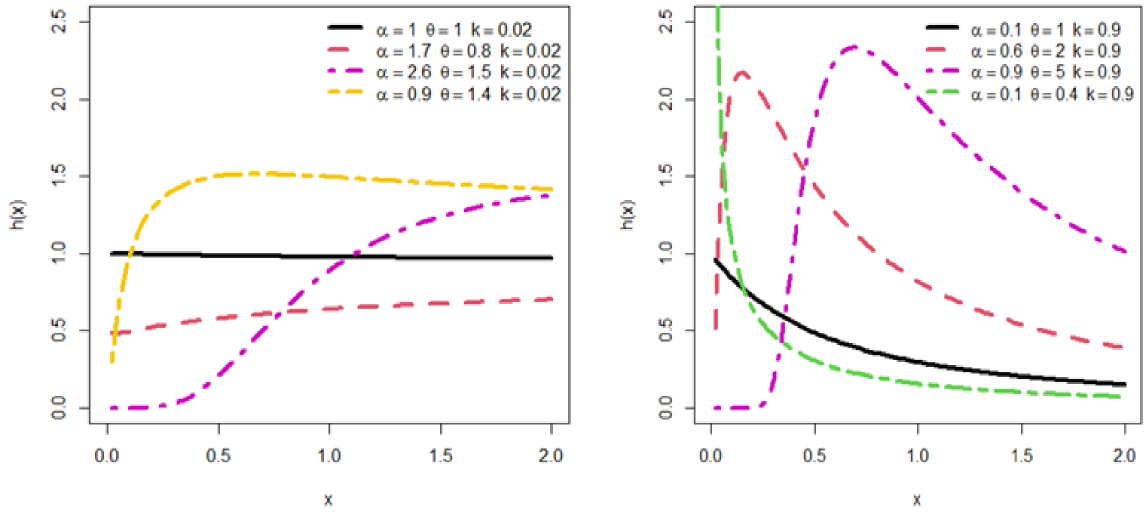

Figure 2: *hrf shapes of EJCA distribution.*

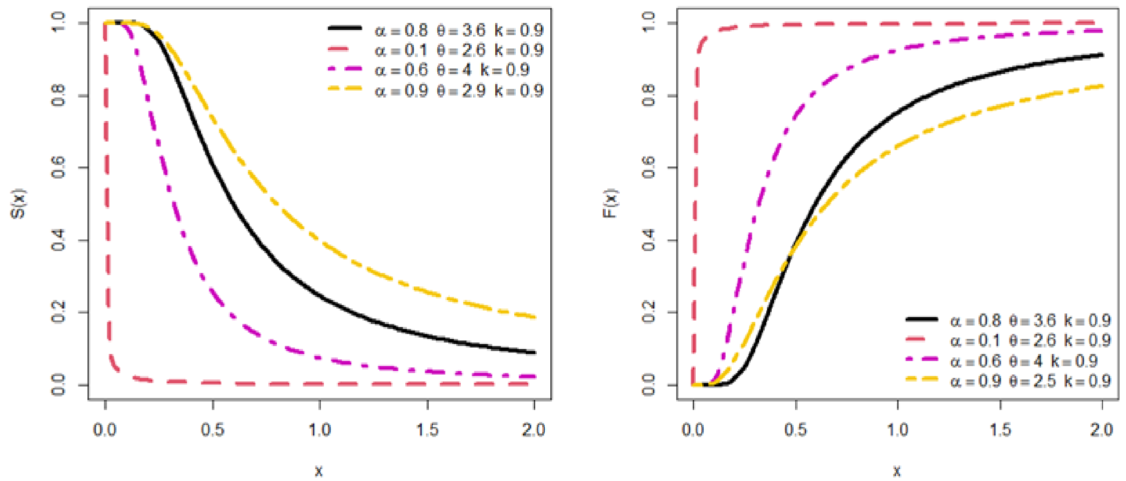

Figure 3: *Shapes of sf and cdf of EJCA distribution with different parameters.*

5. The shape of the pdf and hrf of EJCA Distribution

This section of the article discusses the fundamental mathematical properties of the EJCA distribution. The median, moments, incomplete, mean deviation about the mean, Lorenz curve, Bonferoni curve, Zenga index, quantile function Generalized entropy, the Atkinson index, and the Pietra index are calculated.

5.1. *Quantile Function of the EJCA distribution*

In addition to being a crucial tool for creating random variates, the qf is also necessary to comprehend the median and other positional metrics. Equation (4) can be inverted to obtain the qf as follows.

$$y_q = -\ln\left[1 - e^{\frac{-1}{\alpha}\left[\ln\frac{1}{q}\right]^{\frac{1}{\theta}}}\right]$$

5.2. *Moment-generating function of the EJCA distribution*

Using equation (6), the EJCA distribution's mgf is determined by

$$M_y(t)=E\left(e^{ty}\right)=\int_0^\infty e^{ty}\,\frac{\alpha\theta\left[-\log\left(1-e^y\right)^\alpha\right]^{\theta-1} e^{-\left[-\log\left(1-e^y\right)^\alpha\right]^\theta} e^y}{1-e^y}\,dy$$

$$E\left(e^{ty}\right)=\theta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\sum_{l=0}^{\infty}b_l^{(r)}\frac{(1)}{i!}\int_0^\infty e^{-\left(\frac{j+l}{\alpha}\right)z} z^{\theta i+\theta-1}dy$$

After simplification, we get

$$E\left(e^{ty}\right)=\theta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\sum_{l=0}^{\infty}b_l^{(r)}\frac{(1)}{i!}\frac{\Gamma(\theta i+\theta)}{\left(\frac{r+l}{\alpha}\right)^{\theta i+\theta}}$$

Where $b_l^{(r)}=\sum_{k=0}^{l}b_j^{(m-1)}bl-j, b_k=b_k^{(0)}$

5.3. *The r^th moment of the EJCA distribution*

Using equation (6), the EJCA random variable Y's rth moments' expression is provided by

$$E\left(y^r\right)=\int_0^\infty\frac{y^r\alpha\theta\left[-\log\left(1-e^y\right)^\alpha\right]^{\theta-1} e^{-\left[-\log\left(1-e^y\right)^\alpha\right]^\theta} e^y}{1-e^y}\,dy$$

$$E\left(y^r\right)=\theta\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}b_l^{(r)}\frac{(-1)^i}{i!}\int_0^\infty e^{-\left(\frac{r+l}{\alpha}\right)z} z^{\theta i+\theta-1}dy$$

After simplification, we get

$$E\left(y^r\right)=\theta\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}b_l^{(r)}\frac{(-1)^i}{i!}\frac{\Gamma(\theta i+\theta)}{\left(\frac{r+l}{\alpha}\right)^{\theta i+\theta}},$$

Where $b_l^{(r)}=\sum_{j=0}^{l}b_j^{(m-1)}bl-j, b_j=b_j^{(0)}$

5.4. *Mean deviation about the mean of the EJCA distribution*

The total number of deviations from the mean indicates the degree of scatter in the population to some extent. The mean deviation from the mean can be defined as

$$\delta_1(y)=\int_0^\infty |y-\mu|\, f(y)\,dy$$

Where μ=E(y) denote the mean, respectively.

$$\delta_1(y)=2\left[\mu F(\mu)+\int_0^y yf(y)\,dy\right]$$

The integral can be determined for EJCA distribution as

$$\int_0^y yf(y)\,dy=\int_0^y \frac{y\alpha\theta\left[-\log\left(1-e^y\right)^\alpha\right]^{\theta-1} e^{-\left[-\log\left(1-e^y\right)^\alpha\right]^\theta} e^y}{1-e^y}\,dy$$

After simplification mean deviation from mean is written as,

$$\delta_1(y)=2\left[\mu F(\mu)-\theta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{-i\frac{z}{\alpha}}(-1)^j\left(\frac{i}{\alpha}\right)^{-(\theta i+\theta)}\Gamma\left(\theta i+\theta,\frac{i}{\alpha}z\right)}{i*j!}\right] \qquad (8)$$

5.5. *Lorenz Curve of the EJCA distribution*

In economics, one of the most popular tools for showing how wealth or income is distributed within a population is the Lorenz curve. Its main objective is to illustrate economic inequality so that people, households, and other economic entities can better comprehend the distribution of resources. [18] introduced a curve, Lc, which is defined as

$$L=\frac{\int_0^y yf(y)\,dy}{\mu}$$

By using (8), Solving the above equation, the Lcp obtained for EJCA distribution is

$$L=\frac{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{-i\frac{z}{\alpha}}(-1)^{j}\left(\frac{i}{\alpha}\right)^{-(\theta i+\theta)}\Gamma\left(\theta i+\theta,\frac{i}{\alpha}z\right)}{i*j!}}{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{-i\frac{z}{\alpha}}(-1)^{j}\left(\frac{i}{\alpha}\right)^{-(\theta i+\theta)}\Gamma(\theta i+\theta)}{i*j!}} \tag{9}$$

5.6. *Bonferoni Curve of the EJCA distribution*

[19] intended a partial-means measure of income inequality, which is preferable when the large source of income inequality is the existence of units with significantly lower incomes than others. It is possible to calculate the Bonferroni Curve using the relation,

$$B=\frac{L}{F(y)}$$

Using (9) and (5), we get,

$$B=\frac{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{-i\frac{z}{\alpha}}(-1)^{j}\left(\frac{i}{\alpha}\right)^{-(\theta i+\theta)}\Gamma\left(\theta i+\theta,\frac{i}{\alpha}z\right)}{i*j!}}{e^{-\left[-\log\left(1-e^{y}\right)^{\alpha}\right]^{\theta}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{-i\frac{z}{\alpha}}(-1)^{j}\left(\frac{i}{\alpha}\right)^{-(\theta i+\theta)}\Gamma(\theta i+\theta)}{i*j!}}$$

5.7. *Zenga Index of the EJCA distribution*

[20] and [21] introduced the following income inequality index,

$$Z=1-\frac{\mu_y^{-}}{\mu_y^{+}} \tag{10}$$

Where, $\mu_y^{-}=\frac{\int_0^y yf(y)dy}{F(y)}$, after some algebraic simplification We have,

$$\mu_y^{-}=\frac{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{-i\frac{z}{\alpha}}(-1)^{j}\left(\frac{i}{\alpha}\right)^{-(\theta i+\theta)}\Gamma\left(\theta i+\theta,\frac{i}{\alpha}z\right)}{i*j!}}{e^{-\left[-\log\left(1-e^{y}\right)^{\alpha}\right]^{\theta}}}$$, and μ_y^{+} define as $=\frac{\mu-\int_0^y yf(y)dy}{1-F(y)}$, after

$$\mu_y^{+}=\frac{\mu-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{-i\frac{z}{\alpha}}(-1)^{j}\left(\frac{i}{\alpha}\right)^{-(\theta i+\theta)}\Gamma\left(\theta i+\theta,\frac{i}{\alpha}z\right)}{i*j!}}{1-e^{-\left[-\log\left(1-e^{y}\right)^{\alpha}\right]^{\theta}}}$$

Substituting the value of $\mu_y{}^{+}$ and $\mu_y{}^{-}$ in Eq (10), we get the required result.

5.8. *Pietra Index of the EJCA distribution*

Economic inequality can be measured using the Pietra Index, especially when it comes to the distribution of wealth or income. It belongs to the larger family of inequality measures that evaluate how wealth or income is distributed throughout a population. Some studies use the Pietra Index to provide a different viewpoint on inequality. When researchers wish to quantify inequality with an emphasis on distributional factors, such as poverty or wealth concentration, they occasionally use it. [22] mean deviation is defined as $P_Y = \frac{\delta_1(y)}{2\mu}$, by using (8), we can obtain as, after simplification we get,

$$P_Y = \frac{\left[\mu F(\mu) - \theta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{-i\frac{z}{\alpha}}(-1)^j \left(\frac{i}{\alpha}\right)^{-(\theta i+\theta)} \Gamma\left(\theta i+\theta, \frac{i}{\alpha}z\right)}{i * j!}\right]}{\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{-i\frac{y}{\alpha}}(-1)^j \left(\frac{i}{\alpha}\right)^{-(\theta i+\theta)} \Gamma(\theta i+\theta)}{i * j!}\right]}$$

5.9. *Generalized entropy of the EJCA distribution*

The Generalized Entropy (GE) Index is especially helpful because it can adjust to varying degrees of sensitivity to inequality at various points in the income distribution and generalizes other well-known measures of inequality, such as the Gini coefficient and the Theil index. The GE index was introduced by [23] and [24] is defined as, $G_E = \frac{1}{k(k-1)}\left\{\frac{\mu_r'}{\mu^r} - 1\right\}; r \neq 0,1$,

Where μ_r' is the r[th] moment about origin, after some simplification we have,

$$G_E = \frac{1}{k(k-1)}\left\{\frac{\theta \sum_{i=0}^{\infty}\sum_{l=0}^{\infty} b_l^{(r)} \frac{(-1)^i}{i!} \frac{\Gamma(\theta i+\theta)}{\left(\frac{r+l}{\alpha}\right)^{\theta i+\theta}}}{\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{-i\frac{y}{\alpha}}(-1)^j \left(\frac{i}{\alpha}\right)^{-(\theta i+\theta)} \Gamma(\theta i+\theta)}{i * j!}\right)^r} - 1\right\};$$

5.10. *Estimation Method of the EJCA distribution*

The first four central moments, variance, standard deviation (S.D.), coefficient of skewness (CS), coefficient of kurtosis (CK), and coefficient of variation (CV) for five distinct EJCA models each with a unique set of parameters are shown in Table 1. The distribution for EJCA(2,5,3) has a moderate degree of kurtosis, a positively skewed shape, and a high mean with a major spread. The distribution of EJCA(4,9,0.6), on the other hand, is comparatively flatter and is characterized by a low mean and spread, mild skewness, and low co-efficient of kurtosis (c-k). Despite having a moderate spread, EJCA (4,0.1,7) exhibits considerable skewness and very high kurtosis, indicating a distribution with a prominent peak and heavy tails. With moderate skewness, excessive kurtosis, and a tiny mean and high variance, EJCA (9,–10,6) shows a distribution with outliers and a prominent center peak. Finally, EJCA(8,2,10) shows a highly tight and symmetrical distribution with a very high kurtosis, which suggests that the data is not very variable and that the peak is very sharp.

Table 1:

Analysis of the Variable Y's Moments, Skewness, and Kurtosis for the Selected Parameter Values for EJCA (α, k, θ)

$\mu_k^{/}$	**EJCA (2,5,3)**	**EJCA (4,0.9,6)**	**EJCA (4,0.1,7)**	**EJCA (9,-10,6)**	**EJCA (8,.2,10)**
$\mu_1^{/}$	7.4970	0.8952	1.7529	0.2503	0.1826
$\mu_2^{/}$	1.7223	0.8069	3.1110	0.0627	0.0334
$\mu_3^{/}$	3.8388	0.7326	5.5969	0.0157	0.00612
$\mu_4^{/}$	8.1250	0.6703	10.222	0.0039	0.00112
Variance	1.72238	0.0054	0.0382	4.9329	0.00049
S.D	4.1501	0.0740	0.1954	0.0070	0.00704
Skewness	5.3704	1.0291	1.2695	0.8866	0.75814
Kurtosis	2.7388	128051	38804.98	9681666	270539
CV	5.5357	0.0827	0.1115	0.0280	0.03859

6. A Mixture of EJCA with Exponential (E) distribution

The mixture of two distributions can be defined as

$$f = p\, f_1(y) \ + (1-p) f_2(y) \qquad (11)$$

Here, we proposed cdf and pdf of the Mixture of JCA and E distribution, Let $F_1(y)$ and $f_1(y)$ are the cdf and pdf of JCA distribution.

$$F_1(y) = \mathrm{e}^{-\left[-\log\left[1-\mathrm{e}^{-y(1+y)^{-k}}\right]^{\alpha}\right]^{\theta}} \qquad (12)$$

$$f_1(y) = \frac{\alpha\theta(1+y)^{-k-1}\,[1+(1-k)y]\,\mathrm{e}^{-y(1+y)^{-k}}}{1-\mathrm{e}^{-y(1+y)^{-k}}}\left[-\log(1-\mathrm{e})^{-y(1+y)^{-k}}\right]^{\theta-1}\mathrm{e}^{-\left[-\alpha\log(1-\mathrm{e}^{-y(1+y)^{-k}})\right]^{\theta}} \qquad (13)$$

And $F_2(x)$ and $f_2(x)$ are the cdf and pdf of exponential distribution

$$F_2(y) = 1 - e^{-\mu y} \qquad (14)$$

$$f_2(y) = \mu e^{-\mu y} \qquad (15)$$

Using (13) and (15) in (11), we obtained the pdf of MJE as

$$f = p\frac{\alpha\theta(1+y)^{-k-1}\,[1+(1-k)y]\,e^{-y(1+y)^{-k}}}{1-e^{-y(1+y)^{-k}}}\left[-\log(1-e)^{-y(1+y)^{-k}}\right]^{\theta-1} e^{-\left[-\alpha\log(1-e^{-y(1+y)^{-k}}\right]^{\theta}} + (1-p)\mu e^{-\mu y}) \qquad (16)$$

Using (12) and (14), we obtained the cdf of MJE as

$$F = pe^{-\left[-\log\left[1-e^{-y(1+y)^{-k}}\right]^{\alpha}\right]^{\theta}} + (1-p)(1-e^{-\mu y}) \qquad (17)$$

The family's versatility is demonstrated by the plots in Figures 4, 5 & 6, which show a wide range of density shapes, including decreasing, decreasing-increasing-decreasing, symmetric, and right-skewed distributions. However, the hrf displays constant, bathtub, and upside-down bathtub forms.

From (17), we get the sf as follows,

$$S(y) = 1 - pe^{-\left[-\log\left[1-e^{-y(1+y)^{-k}}\right]^{\alpha}\right]^{\theta}} - (1-p)(1-e^{-\mu y}) \qquad (18)$$

The function of the failure or hazard rate function, $h(x)$ is obtained from (16) and (18) as,

$$h(y) = \frac{p\frac{\alpha\theta(1+y)^{-k-1}\,[1+(1-k)y]\,e^{-y(1+y)^{-k}}}{1-e^{-y(1+y)^{-k}}}\left[-\log(1-e)^{-y(1+y)^{-k}}\right]^{\theta-1} e^{-\left[-\alpha\log(1-e^{-y(1+y)^{-k}}\right]^{\theta}} + (1-p)\mu e^{-\mu y})}{1-pe^{-\left[-\log\left[1-e^{-y(1+y)^{-k}}\right]^{\alpha}\right]^{\theta}} - (1-p)(1-e^{-\mu y})} \qquad (19)$$

From (18), the reversed hazard rate function is obtained as

$$r(y) = \frac{p\frac{\alpha\theta(1+y)^{-k-1}\,[1+(1-k)y]\,e^{-y(1+y)^{-k}}}{1-e^{-y(1+y)^{-k}}}\left[-\log(1-e)^{-y(1+y)^{-k}}\right]^{\theta-1} e^{-\left[-\alpha\log(1-e^{-y(1+y)^{-k}}\right]^{\theta}} + (1-p)\mu e^{-\mu y})}{pe^{-\left[-\log\left[1-e^{-y(1+y)^{-k}}\right]^{\alpha}\right]^{\theta}} + (1-p)(1-e^{-\mu y})}$$

The cumulative hazard rate function is defined as $H(y) = -logS(y)$, from (18), we get

$$H(y) = -\log(1 - pe^{-\left[-\log\left[1-e^{-y(1+y)^{-k}}\right]^{\alpha}\right]^{\theta}} - (1-p)(1-e^{-\mu y}))$$

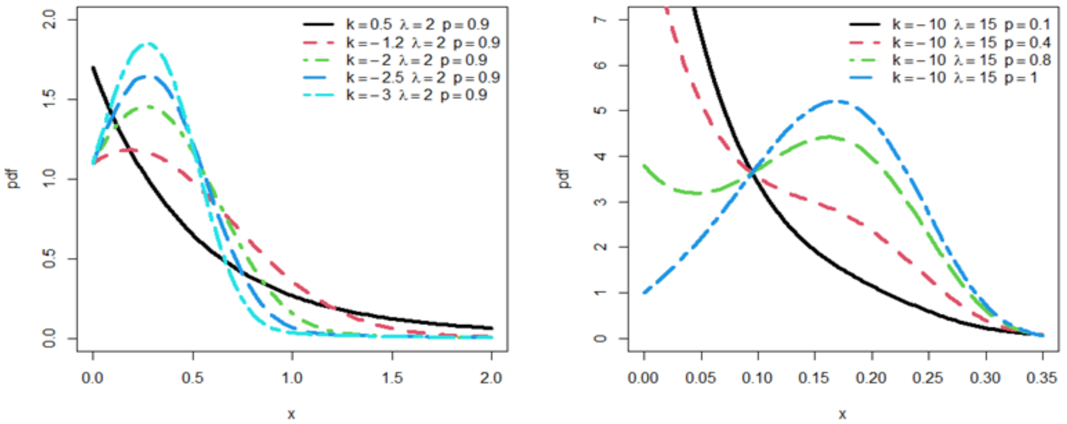

Figure 4: Plots of the MJE distribution in PDF.

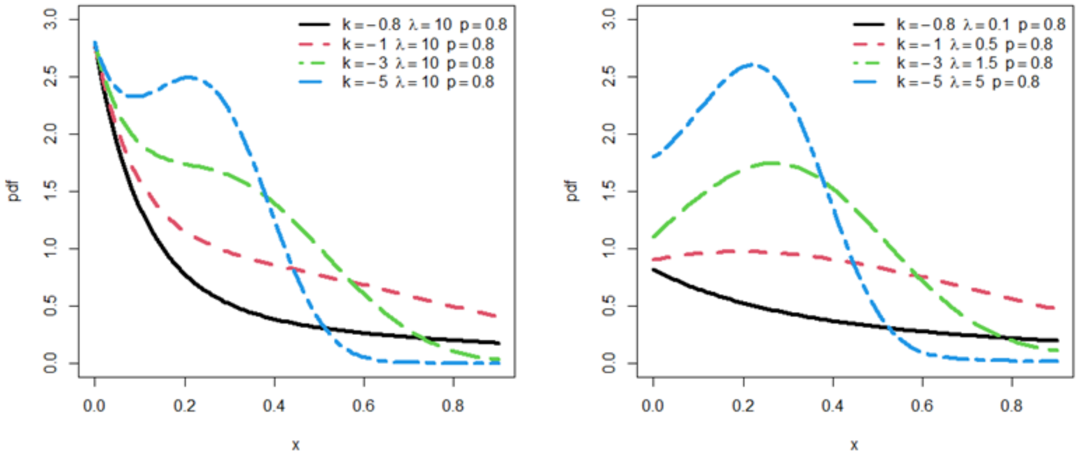

Figure 5: The MJE plots for the PDF.

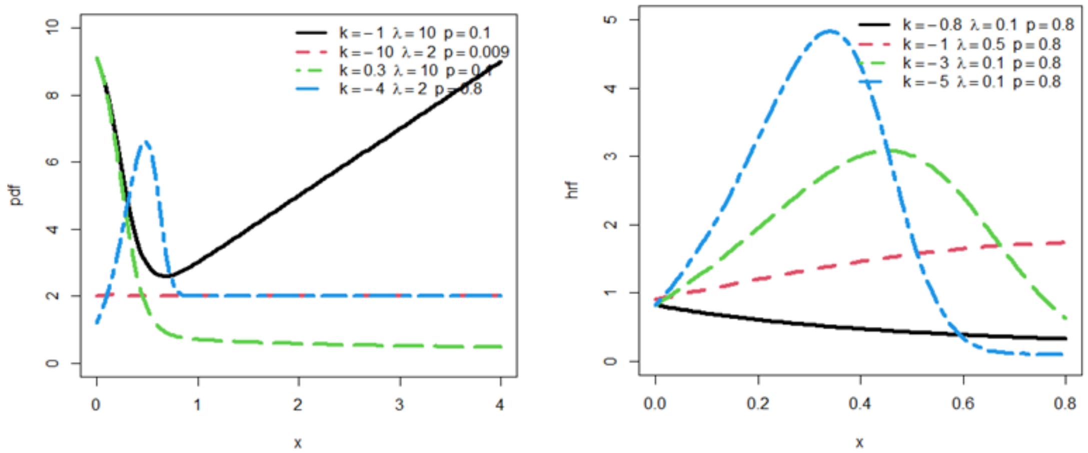

Figure 6: The plots for HRF of MJE.

7. Applications on real data sets

The applicability of the suggested distribution to actual datasets is carefully examined in this section. We illustrate how the distribution offers a better fit than traditional models and successfully captures intricate patterns. With the help of these case studies, we demonstrate the distribution's adaptability to a range of domains, demonstrating its capacity to represent complex relationships and produce more precise predictions in real-world situations.

7.1. *Applications of the JCA distribution*

We demonstrate the importance of the EJCA distribution in this section by presenting two applications of actual datasets. The JCA, Exponentiated Burr Distribution (EB), Exponentiated Weibull Distribution (EWD), Weibull Distribution (WD), and the EJCA distribution are among the models with which we compare the EJCA distribution. The Kolmogorov-Smirnov (K-S) goodness-of-fit test is used to evaluate the validity of the fitted models for each dataset, and the p-values in each instance show that the models fit the data very well. Through the computation of MLE's and their standard errors, we further investigate the application of the New Exponentiated Burr Type III model to two real-life phenomena.

The Cramér-von Mises (W), Anderson-Darling (A), and Akaike Information Criterion (AIC) tests, among others, all show that the lower statistics, except the higher p-values in the K-S test, indicate good model fits. The simulated annealing method (SAAN(S)) is used as a global optimization algorithm to determine initial values, and BFGH (B) yields the optimal solution. The required calculations are performed using the R programming language. [24] is the source of Dataset I, and [25] reports on uncensored data on remission times (in months) from a random sample of 128 patients with bladder cancer.

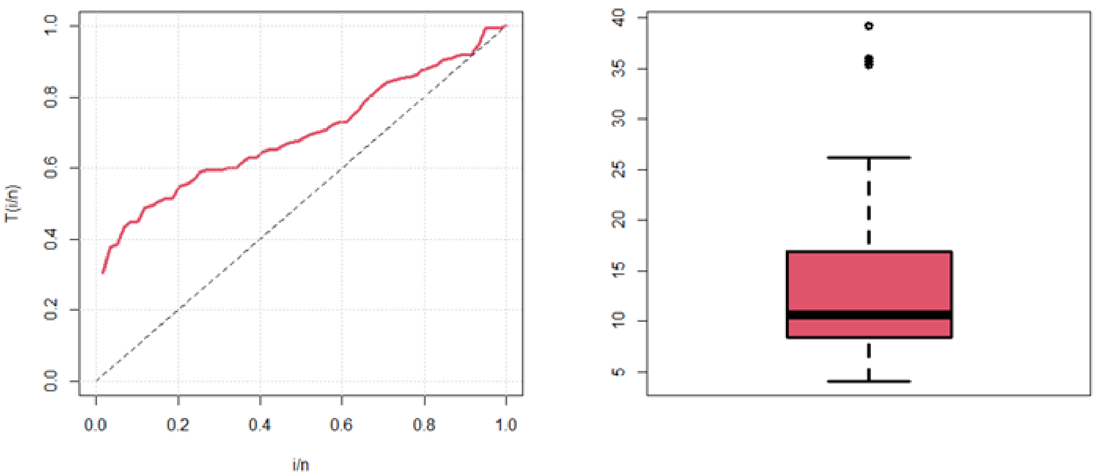

Figure 7: Analysis using a Box Plot and a TTT Plot for the 1st data.

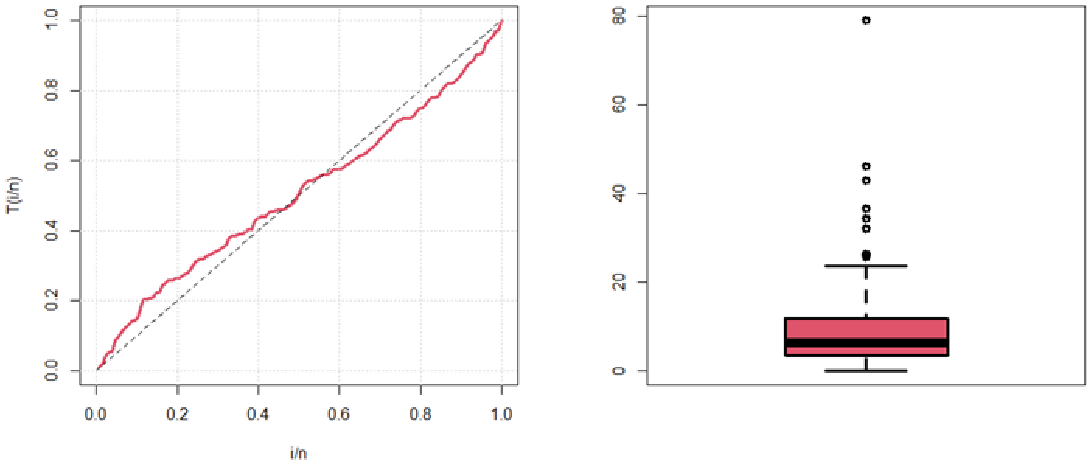

Figure 8: TTT and Box Plot of the 2nd Data.

A box plot for Dataset 1 and a TTT (Total Time on Test) plot showing an increasing trend is shown in Figure 7. The box plot summarizes the distribution of the dataset, including the median, quartiles, and possible outliers, while the TTT plot shows a steady increase over time. A TTT plot, shown in Figure 8, shows a pattern of both increasing and decreasing trends over time. This plot, which shows periods of growth followed by decline, sheds light on how the data changes over time. A box plot for Dataset II is also included in the figure. When taken as a whole, these visualizations contribute to a thorough comprehension of the behaviour and general distribution trends of the dataset.

Figure 9 & 10 show a violin plot that combines important statistical characteristics into a single, unified shape to visually depict a dataset's distribution. The median, which represents the middle value of the data, is indicated by a thin line at its centre. The interquartile range (IQR), which covers the 25th to 75th percentiles and indicates where the majority of the data is located, is represented by a thicker section surrounding this line. A kernel density estimation is reflected in the violin's curved shape, which shows how frequently values occur across the range. Narrower sections suggest fewer observations, while wider sections indicate higher density (more data points). Since these components work together to give a thorough understanding of the data's distribution, central tendency, and probability density, violin plots are especially helpful when comparing distributions across categories or spotting multimodality. The plots also provide additional information about the underlying distributions of datasets I and II by displaying the kernel density fit for these datasets.

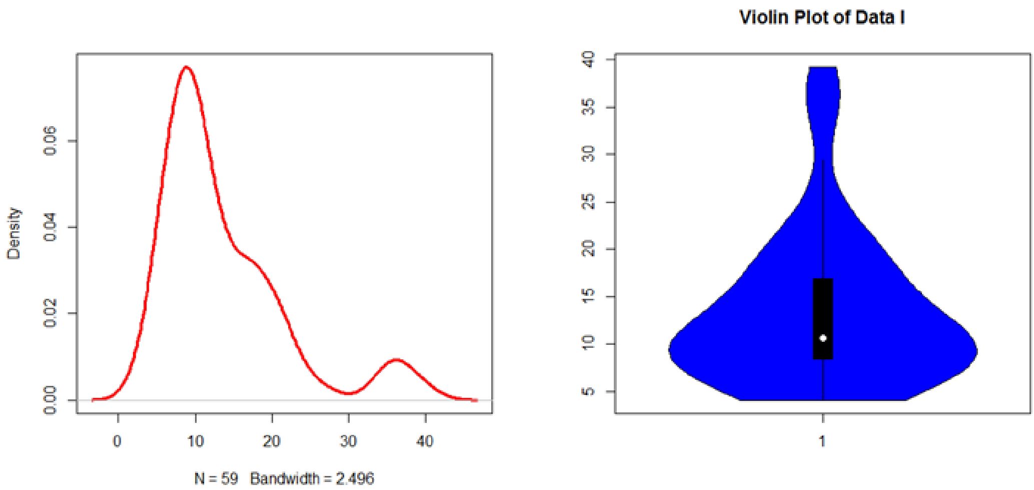

Figure 9: *Kernal density and violin plot for the data set I.*

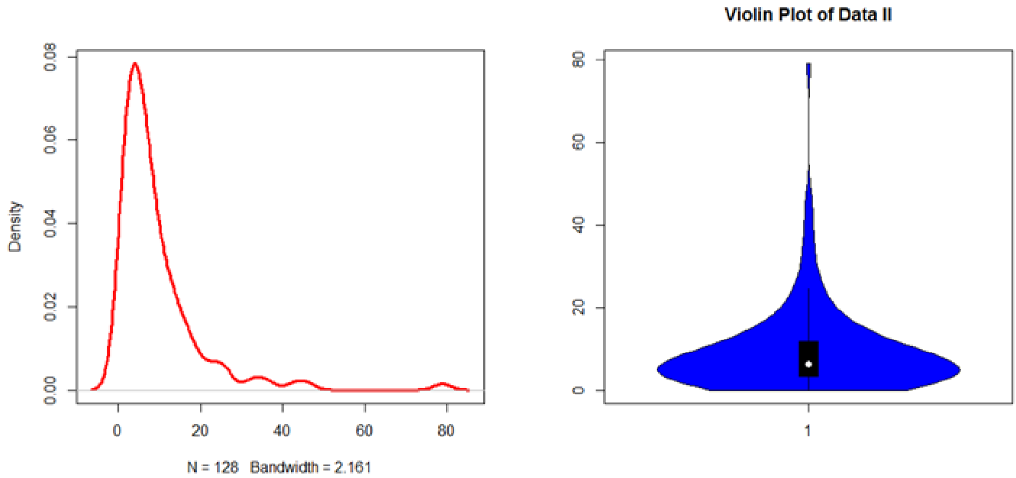

Figure 10: *Kernal density and violin plot for the data set II.*

Table 2:

Statistical Evaluation of Failure Time Data

Data Sets	*n*	Min.	Mean	Median	s.d.	Skewness	Kurtosis	Max	Range	S.E
I	59	4.1	13.49	10.60	8.05	1.57	2.08	39.2	35.1	1.05
II	128	0.08	9.37	6.39	10.51	3.25	15.2	79.05	78.97	0.93

Table 2 shows that the 59 observations in Dataset I have a minimum value of 4.1 and a maximum value of 39.2. The distribution is strongly right-skewed with a skewness of 1.57, the median is 10.60, the standard deviation is 8.05, and the kurtosis of 2.08 suggests a shape that is similar to a normal distribution. There are 128 observations in Dataset II. Significant data variability is indicated by the median value of 6.39 and the standard deviation of 10.51. The distribution is significantly skewed to the right (skewness = 3.25), with a lengthy tail extending toward higher values and the majority of values grouped at the lower end. Extreme leptokurtosis (kurtosis = 15.2), which indicates abnormally heavy tails in comparison to a normal distribution, is present along with this asymmetry.

Table 3:

The failure time dataset's estimated MLE's, standard errors, and corresponding confidence intervals are shown in parentheses

Models	$\hat{\alpha}$	$\hat{\theta}$	$\hat{k}$	$\hat{\lambda}$	$\hat{\mu}$
EJCA	8.4894	2.1387	0.6208	-------	-------
	(0.7629)	(1.4833)	(2.0085)		
JCA	------	-------	0.7078	-------	-------
			(0.0284)		
EWD	0.7771	-------	-------	3.1447	10.6868
	(0.1409)			(1.5117)	(5.8527)
WD	0.0068	-------	-------	1.8274	-------
	(0.0027)			(0.1330)	
BD	2.6501	0.81664	120.4713	-------	-------
	(6.4347)	(2.0164)	(0.4679)		

Using the EJCA, JCA, EWD, WD, and EB models, the estimated density and CDF plots for the first dataset are displayed in Figure 11. As anticipated, the CDFs are non-decreasing, however the density plots show a decreasing and right-skewed shape. Figure 12 shows the plots for the second dataset, where the CDFs do not decrease and the estimated densities show a decreasing pattern.

Table 4:

The Log-likelihood, AIC, BIC, CAIC, HQIC, A, W and KS (p-value) values for the failure time data set

Models	AIC	BIC	CAIC	HQIC	A	W	KS (p-value)
EJCA	382.3535	388.5862	382.7899	384.7865	0.0412	0.2523	0.0624 (0.9754)
JCA	568.8081	570.8857	568.8783	569.6191	0.0997	0.5886	0.7408 (0.0000)
EWD	385.9994	392.2320	386.4357	388.4323	0.6596	0.1116	0.0941
(0.6719)							
WD	398.587	402.7420	398.8012	400.2089	1.8532	0.2905	0.1403

							(0.1957)
EB	384.0296	390.2622	384.4660	386.4626	0.0345	0.2514	0.0620 (0.9700)

Table 5:

MLEs, standard errors, confidence intervals (in parentheses) values for data set II.

Models	$\hat{\alpha}$	$\hat{\theta}$	$\hat{k}$	$\hat{\lambda}$	$\hat{\mu}$
EJCA	9.8714 (3.4625)	0.5686 (0.0990)	0.3857 (0.0585)	-------	-------
JCA	------	-------	0.6730 (0.0233)	-------	-------
EWD	0.2207 (0.00791)	-------	-------	0.0047 (0.0013)	80.2883 (17.0563)
WD	0.0939 (0.0190)	-------	-------	1.0477 (0.0675)	-------
BD	0.6469 (0.1591)	1.9291 (0.6113)	9.3878 (4.6095)	-------	-------

Table 6:

Log-likelihood, HQIC, A, W, KS (p-value), BIC, CAIC, and AIC values for data set II

Models	AIC	BIC	CAIC	HQIC	A	W	KS (p-value)
EJCA	826.7372	835.2932	826.9307	830.2135	0.2477	0.0388	0.0424 (0.9750)
JCA	987.3266	990.1787	987.3584	988.4854	1.2203	0.1865	0.5077 (0.0000)
EWD	849.0946	857.6507	849.2881	852.5710	1.7180	0.2651	0.0762 (0.4460)
WD	832.1738	837.8778	832.2698	834.4913	0.7863	0.1313	0.0699 (0.5573)
EB	855.5399	864.0960	855.7335	859.0163	2.1458	0.3340	0.1014 (0.1438)

Several metrics are used to compare how well these distributions fit, such as the Bayesian information criterion (BIC), the corrected AIC (CAIC), the Hannan–Quinn information criterion (HQIC), the Kolmogorov–Smirnov (KS) statistics, its p-value (KS p-value), the Anderson–Darling (A∗), and the Cram´er-von Mises (W∗) statistics. The outcomes of these measures for both data sets are shown in Tables 4 & 6. Additionally, they provide the standard errors (SEs) and MLEs of the parameters of the fitted distributions for both data sets in Table 3 & 5. The EJCA distribution has the lowest values for all goodness-of-fit metrics, according to the figures in these tables.

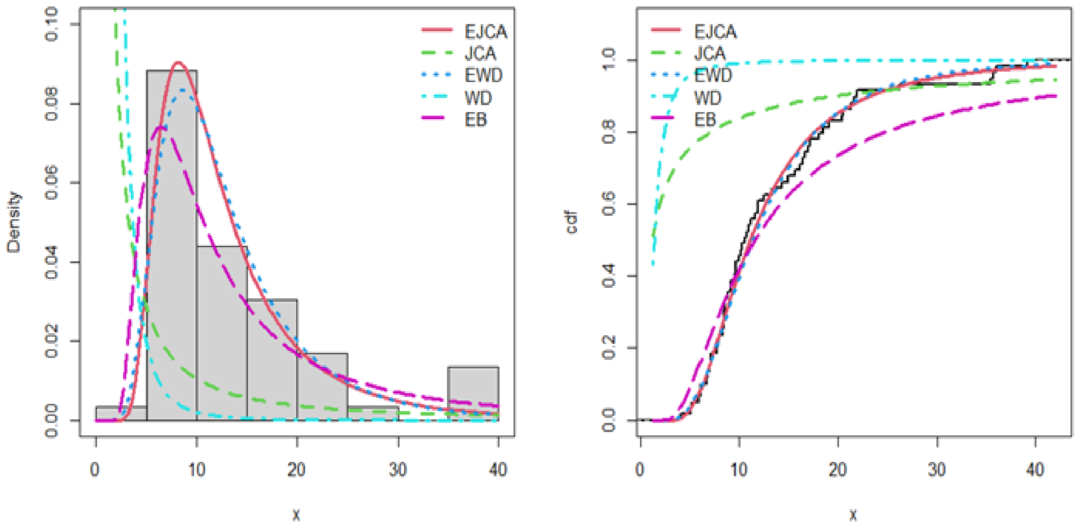

Figure 11: Plots for the epdf, ecdf, of the EBII model for the I data set.

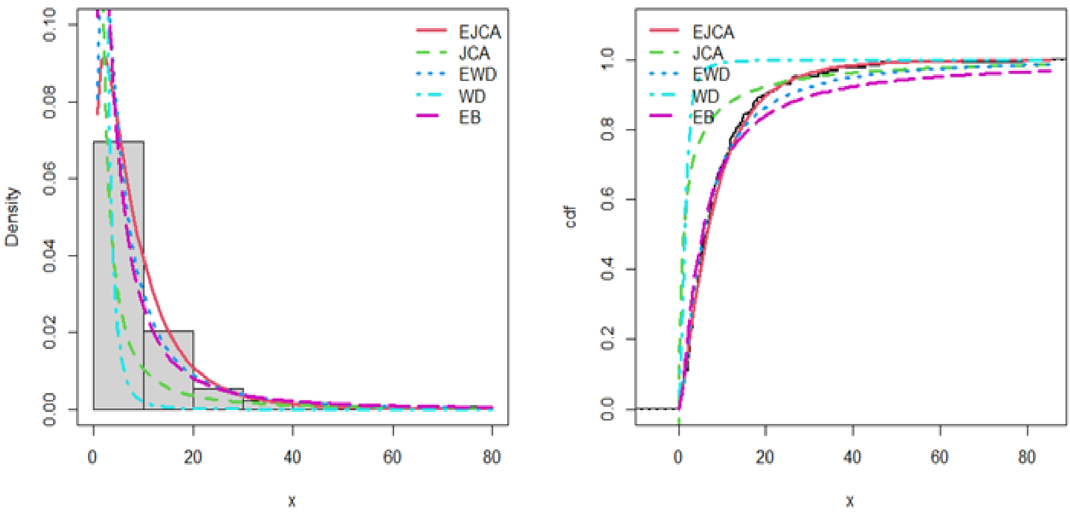

Figure 12: *The Plots for the Epdf, Ecdf, of the EBII model for the II data set*.

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7.2. *Applications for the MJE distribution*

The 'I' dataset was collected with certain parameters, such as a 2 mm sheet thickness and a 9 mm hole diameter [26]. In contrast, the 'D2' dataset contains the survival durations (in weeks) of 33 pa-tients who died of Acute Myelogenous Leukemia, a subject that was recently studied by [27].

Table 7:

MLEs, standard errors, (in parentheses), A, W and KS (p-value) for data set I

Models	$\hat{k}$	$\hat{\mu}$	$\hat{p}$	A	W	KS (*p*-value)
MJE	-9.6034 (0.7067)	14.7046 (1.2410)	0.9002 (0.1141)	0.6265	0.0974	0.1172 (0.4978)
J	-9.724874 (0.6696)	------	------	0.6604	0.1035	0.1396 (0.2836)

E	6.5789 (0.9304)	------	------	1.7859	0.3240	0.2859 (0.0005)

Table 8:

MLEs, standard errors, (in parentheses), A, W and KS (p-value) for data set II

Models	$\hat{k}$	$\hat{\mu}$	$\hat{p}$	**A**	**W**	**KS (*p*-value)**
MJE	0.4460 (0.0046)	0.0214 (0.3156)	0.1309 (0.1169)	0.6040	0.0860	0.1323 (0.6102)
J	0.7869 (0.0285)	------	------	0.6863	0.1100	0.5440 (0.0000)
E	0.0244 (0.0042)	------	------	0.6729	0.0972	0.2182 (0.0863)

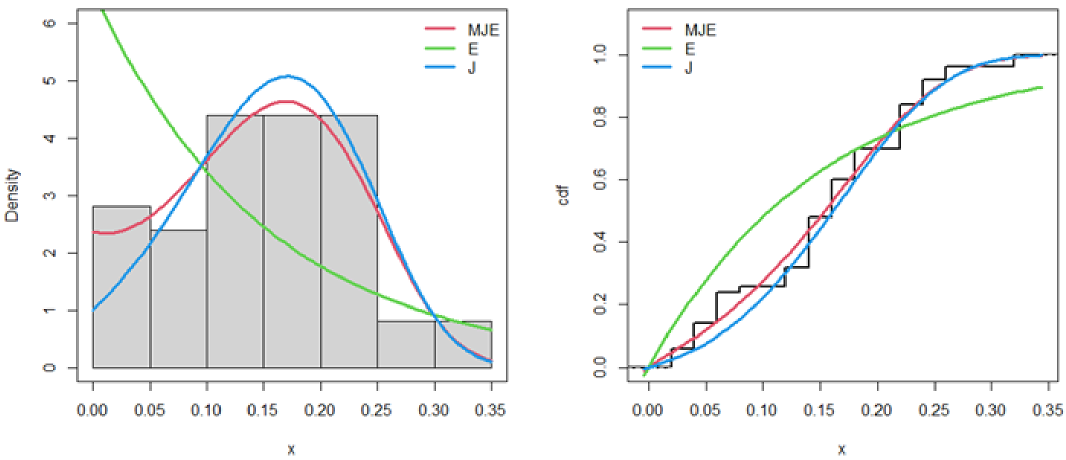

Figure 13: The plots for estimated pdf and CDF of MJE for data set I.

Figures 4, 5 & 6 display the MJE model's pdf and HRF shapes for a few selected parameter values. Plots of the HRF, bathtub, decrease, unimodal, reversed-J shaped, J shaped, decreasing-increasing-decreasing hazard rates and the PDF are shown to be symmetrical, left and right skewed, and highly flexible. The estimated density and CDF plots for the first and second datasets are shown in Figures 13 & 14. As expected, the CDF plots are non-decreasing, while the density plots are symmetric and decreasing.

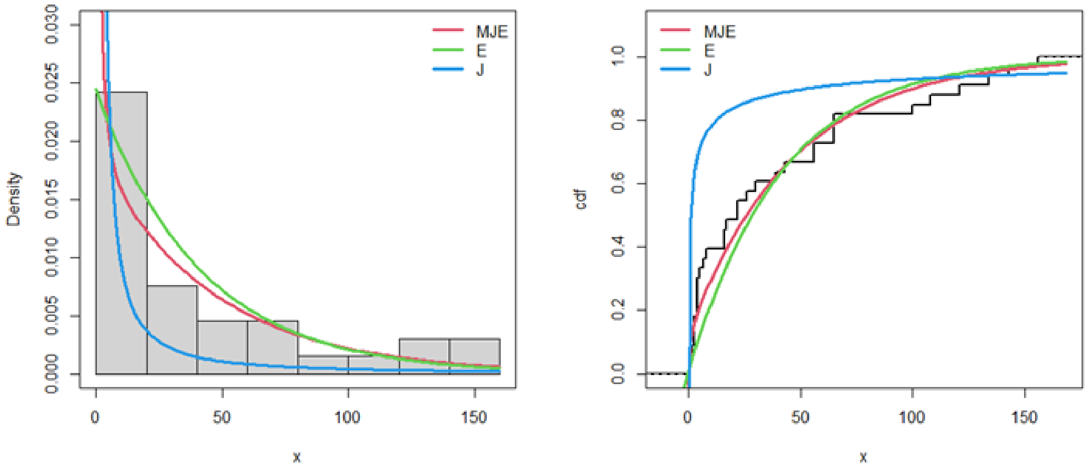

Figure 14: *The plots for estimated pdf and cdf of MJE for data set II.*

The metrics used to compare how well these distributions fit include the Bayesian information criterion (BIC), Hannan–Quinn information criterion (HQIC), Akaike information criterion (AIC), consistent AIC (CAIC), Kolmogorov–Smirnov (KS) statistics, its p-value (KS p-value), Anderson–Darling (A∗), and Cram´er-von Mises (W∗) statistics. Tables 7 & 8 display the results of these measurements for both sets of MJE distribution data. For both data sets, they also offer the MLEs and standard errors (SEs) of the parameters of the fitted distributions. The figures in these tables show that the MJE distribution has the lowest values for all goodness-of-fit metrics.

8. Conclusion

The EJCA distribution is a more adaptable and versatile generalization of the JCA distribution that we introduced in this study to improve its suitability for statistical modelling. The EJCA distribution exhibits enhanced adaptability by adding new methods, which makes it a useful resource for researchers in a variety of disciplines. Additionally, we created a mixed model by the JCA distribution with the classical exponential distribution, creating a new hybrid model that combines the advantages of both parent distributions.

To verify the practical usefulness of the suggested models, we carried out comprehensive empirical analyses using four real-world datasets. The results showed that the EJCA and mixed models outperform competing distributions in terms of fit and flexibility, demonstrating their effectiveness in modelling diverse data scenarios. The EJCA distribution's quantile function, moments, reliability metrics, and maximum likelihood estimators were all mathematically determined with precision. Additionally, mixture representations were investigated to provide deeper insights into their structural behaviour.

Overall, this research adds to the expanding body of flexible statistical distributions, offering reliable alternatives for modelling and inference in a variety of scientific domains. The successful application of these models to real data highlights their potential in statistical practice, providing researchers with enhanced tools for data analysis. Future work may explore Bayesian estimation, regression modelling, and further extensions to accommodate more complex data structures.

Declaration

Author Contribution: ***M.S*** served as the corresponding author of the paper and provided valuable support to the student during the writing process. She assisted in organizing the manuscript and ensuring that the research was clearly and effectively communicated. ***F.J*** contributed significantly to the development and refinement of the core idea for the paper. He also played a key role in writing and rewriting the content to enhance clarity and improve the overall quality of expression. Additionally, he was responsible for proofreading and polishing the final version to ensure accuracy, coherence, and a professional standard of presentation. ***Z.H*** provided the original idea for the project and offered valuable guidance throughout its development. He also assisted in writing and running the codes for practical applications of the proposed model to real data, and ensured that all necessary components were accurately completed and effectively presented. ***S.I*** played a key role in deriving the mathematical properties of the proposed distribution. He also contributed to the writing process by helping structure the content and present the technical details in the manuscript. ***R.A*** contributed to the paper by conducting a comprehensive literature review to establish the context and relevance of the study. She also played an active role in drafting sections of the manuscript and performed thorough proofreading to ensure clarity, coherence, and academic quality

Conflict of Study: *NA*

Ethical Approval and Consent of Participation: *NA*

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