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Power Generalized KM-Transformation for Non-Monotone Failure Rate Distribution

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ABSTRACT

A more useful transformation model, KM Transformation, for reliability and lifetime data analysis is introduced by Kavya & Manoharan (2021). Power generalization technique is the best approach for analysing a parallel system. In this article, we present a new transformation called Power Generalized KM-Transformation (PGKM) to obtain a more appropriate model while monotone and non-monotone behaviour for the failure rate function occurs. We derived the moments, moment generating function, characteristic function, quantiles, etc. for the PGKM transformation of Exponential distribution (PGKME). Distributions of minimum and maximum are obtained. Estimation of parameters of the PGKME distribution is performed via maximum likelihood method, method of moment, and least square estimation method. A simulation study is performed to validate the maximum likelihood estimator (MLE). Analysis of three sets of real data is provided.

Keywords: KM-Transformation; PGKM Transformation; Exponential distribution; Bathtub shaped failure rate; Reliability

1. Introduction

Lifetime models with monotone and non-monotone failure rate functions have wide applications in engineering and lifetime data analysis. Statistical distributions are widely used in reliability and lifetime data analysis. Existing models have limitation in respect of fitness to the given data. It would be nice if there was a better choice than the existing models. So, we can do more accurate reliability analysis, stress-strength analysis, etc. The use of the exponential distribution is sometimes inappropriate because its failure rate is constant always. Weibull, Gamma, Lindley, and generalized Lindley distributions are the examples of distributions with non-monotone failure rates. In engineering, when we consider a parallel system, in which each of the components are distributed as any well known distributions, we must consider power transformation of the component distribution. Since KM transformation, introduced by Kavya and Manoharan(2021), is found to be a better choice for lifetime data analysis than existing distributions, its use in parallel system has to be explored in detail. In biological and engineering sciences, the study of the life span of organisms, systems or devices, materials, etc. is of great importance. Actual physical considerations of the failure mechanism may often lead to a specific distribution, but sometimes, the choice is made based on how well the actual failure time data fit the distribution. In lifetime studies, the shape and homogeneity of the failure rate function are examined, and the search for plausible candidates is conducted in the various available distributions.

Gupta & Kundu (1999) introduced the generalized exponential distribution to model parallel systems with exponential components. Exponential distribution is the failure time distribution of a component or a system when failure occurs only due to random shocks or jolts or over voltage etc. For the failure of a component or system, the cause of the failure could be gradual degradation, random failures, failure upon installation, etc.

The upside-down bathtub-shaped failure rate models are widely used in biological and reliability studies. For example, a failure rate curve can be seen over the course of a disease whose death rate peaks after a certain period of time and gradually declines. Lifetime models that display hazard rates in the shape of an upside-down bathtub find their application in survival analysis. For a study of head and neck cancer data, see Efron (1988), data on 3878 breast carcinoma cases seen in Edinburgh from 1954 to 1964, showed the use of upside-down bathtub failure rate models. In reliability theory, examples of upside-down bathtub-shaped models can be found in accelerated life testing. Recently Alkami (2015), Deepthi & Chacko (2020), Dimitrakopoulou et al. (2007), Kavya & Manoharan (2020), Maurya et al. (2017) and Sharma et al. (2014), etc have developed and studied upside down bathtub-shaped failure rate distributions. The major objective of this paper is the development of parsimonious models used in the modelling and analysis of lifetime data. However, many families of distributions used in stochastic modelling of lifetime data are often non-parsimonious, unnatural, theoretically unjustified, and sometimes redundant. For the present study we consider the KM transformation and their use in parallel system. Recently, Gauthami and Chacko (2025) discussed power generalized transformation of inverse weibull distribution. Emphasizing this transformation, which effectively extends the DUS model through exponentiation, recent work has expanded this approach to the Weibull (PGDUS-W) and Lomax (PGDUS-L) distributions (see Thomas and Chacko(2023)) and the Inverse Kumaraswamy (PGDUSIK) (see Amrutha and Chacko(2024)) distribution.

In KM transformation, one can model lifetime data more accurately than standard lifetime distributions such as exponential, Weibull, and Lomax. In a parallel system, if the components are distributed as the KM transformation of some baseline model, one has to take PGKM transformed model. A detailed study of the PGKM transformation for lifetime distributions needs to investigate the applicability.

The distributions presented using the KM transformation are parsimonious, no new parameters are used, and show a monotone failure rate. If this distribution were applied to the lifetime data of the components in a parallel system, we would get a better model for the parallel system. Therefore, more accurate reliability models would have been created based on the choice of baseline models. The advantage of using the PGKM transformation is that the estimation of the parameters will be easier because the number of parameters will not increase in the KM transformation, but the fitness level will increase.

Let Y be a random variable with cumulative distribution function (CDF) $F(y)$ and probability density function (PDF) $f(y)$ of some baseline distribution. Then the cdf $G(y)$ of new distributionis defined as,

$$G(y) = \frac{e}{e-1}\left(1 - e^{-F(y)}\right).$$

the corresponding pdf is given by

$$g(y) = \frac{e}{e-1} f(y) e^{-F(y)}.$$

We consider Exponential distribution as baseline distribution for the KM transformation.

This paper is organized as follows. In Section 2, the power generalized KM transformation to existing distributions is presented. Also, a new distribution using the Exponential distribution as the base distribution in the Power Generalized KM transformation is introduced. Different forms of the failure rate function for different values the parameters are given in Section 3. In Section 4, moments, moment generating function and characteristic are derived. The quantile function is discussed in Section 5. Mean residual life function and stochastic ordering are discussed in Sections 6 and 7, respectively. In Section 8, the distribution of maximum and minimum order statistics are discussed. Various estimation techniques are explained in Section 9. In Section 10, a simulation study is conducted to compare the performances of the proposed estimators. In Section 11, a comparison of various models is made with the help of three sets of real data. Finally, conclusions are given in Section 12.

2. Power Generalized KM-Exponential Distribution

In this section, we introduce a new transformation, Power generalized KM transformation and study the new lifetime model Power generalized KM transformation using Exponential distribution as the baseline distribution. Kavya and Manoharan(2021) introduced a transformation, named as KM-Transformation and is defined as

$$G(y) = \frac{e}{e-1}\left(1 - e^{-F(y)}\right) \quad (1)$$

where $F(y)$ is the cdf of some baseline distribution. If we consider a parallel system in which components are distributed as KM transformation of some existing distributions. The above transformation is generalized with a new shape parameter α, given that α is always greater than zero. Therefore, cdf of Power Generalized KM-transformation is defined as

$$G(y) = \frac{\left[e\left(1-e^{-F(y)}\right)\right]^{\alpha}}{(e-1)^{\alpha}} \tag{2}$$

Then the pdf and failure rate functions are respectively obtained as

$$g(y) = \frac{\alpha e^{\alpha}}{(e-1)^{\alpha}} f(y) e^{-F(y)} \left(1 - e^{-F(y)}\right)^{\alpha-1} \tag{3}$$

and

$$r(y) = \frac{\alpha e^{\alpha} f(y) e^{-F(y)} \left(1-e^{-F(y)}\right)^{\alpha-1}}{(e-1)^{\alpha} - \left(e\left(1-e^{-F(y)}\right)\right)^{\alpha}}. \tag{4}$$

Power Generalized KM-Transformation to the baseline Exponential distribution is considered. The reason for choosing the Exponential distribution is that it has wide application in reliability theory for modelling data due to random failures. Substitute the cdf of exponential distribution $F(y) = 1 - e^{-\lambda y}, y > 0, \lambda > 0$ in Eq. (2) to get the new distribution and it is called Power Generalized KM-Exponential (PGKME) distribution. The pdf and cdf of the new distribution is derived as,

$$g(y) = \frac{\alpha \lambda e^{\alpha}}{(e-1)^{\alpha}} e^{-\left(1+\lambda y - e^{-\lambda y}\right)} \left(1 - e^{-\left(1-e^{-\lambda y}\right)}\right)^{\alpha-1} \tag{5}$$

and

$$G(y) = \frac{\left[e\left(1-e^{-\left(1-e^{-\lambda y}\right)}\right)\right]^{\alpha}}{(e-1)^{\alpha}} \tag{6}$$

The survival function of PGKME distribution is

$$S(y) = 1 - \frac{\left[e\left(1-e^{-\left(1-e^{-\lambda y}\right)}\right)\right]^{\alpha}}{(e-1)^{\alpha}} \tag{7}$$

and the failure rate function is

$$r(y) = \frac{\alpha \lambda e^{\alpha} e^{-\left(1+\lambda y-e^{-\lambda y}\right)} \left[1-e^{-\left(1-e^{-\lambda y}\right)}\right]^{\alpha-1}}{(e-1)^{\alpha} - \left(e\left(1-e^{-\left(1-e^{-\lambda y}\right)}\right)\right)^{\alpha}} \tag{8}$$

and the PGKME distribution is denoted as PGKME(α, λ) distribution.

3. Shape of The Distribution and Hazard Rates

The shape of the distribution is an important characteristic because it gives an idea of the nature of the distribution, like skewness, kurtosis etc. The plots of pdf for different values of α and λ using equation Eq. (5) are given in Figure 1.

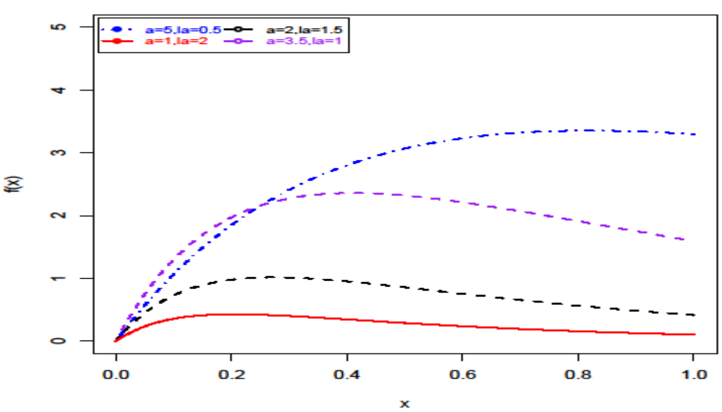


Fig 1: The probability density function of PGKME(α, λ)

To study about the shapes of hazard rate, Glaser's technique can be used, Glaser (1980). Let

$\psi(u) = \frac{-g'(u)}{g(u)}$ where $g(u)$ is the pdf and $g'(u)$ is first derivative of $g(u)$ with respect to u. Then

Theorem 3.1

1. *If* $\psi'(u) > 0$ *for all* $u > 0$, *then distribution has increasing failure rate.*
2. *If* $\psi'(u) < 0$ *for all* $u > 0$, *then distribution has decreasing failure rate.*
3. *Suppose there exits* $u^* > 0$ *such that* $\psi'(u) < 0$, *for all* $u \in (0, u^*)$, $\psi'(u^*) = 0$ *and* $\psi'(u) > 0$ *for all* $u > u^*$ *and* $\epsilon = \lim_{u\to 0} g(u)$ *exists. Then if*

a. $\epsilon = \infty$, *distribution has bathtub shaped failure rate.*

b. $\epsilon = 0$, *distribution has increasing failure rate.*

For the proposed distribution,

$$\psi(u) = \lambda(1 + e^{-u}) - \lambda(\alpha - 1)e^{-(1+\lambda u - e^{-\lambda u})}\left(1 - e^{-(1-e^{-\lambda u})}\right)^{-1},$$

and

$$\psi'(u) = -\lambda^2 e^{-\lambda u} + \lambda^2(\alpha - 1)e^{-(1+\lambda u - e^{-\lambda u})}\left(1 - e^{-(1-e^{-\lambda u})}\right)^{-2}\left(1 + e^{-\lambda u} - e^{-(1-e^{-(\lambda u)})}\right). \quad (9)$$

It can easily be checked that the following three cases arise:

1. When$\alpha > 1, \psi'(u) > 0$ for all $u > 0$, hence the distribution has increasing failure rate.
2. When $\alpha \leq 1$,$\psi'(u) < 0$for all $u > 0$, hence the distribution has decreasing failure rate.
3. When$0.5 < \alpha \leq 1$, then we verified that there always exists a u^* such that $\psi'(u) < 0$ when $u \in (0, u^*)$, and $\psi'(u^*) = 0$ and $\psi'(u) > 0$ for all $u > u^*$, where u^* depends on the value of α and λ but the exact functional form of u^* in term of α and λ could not be derived.

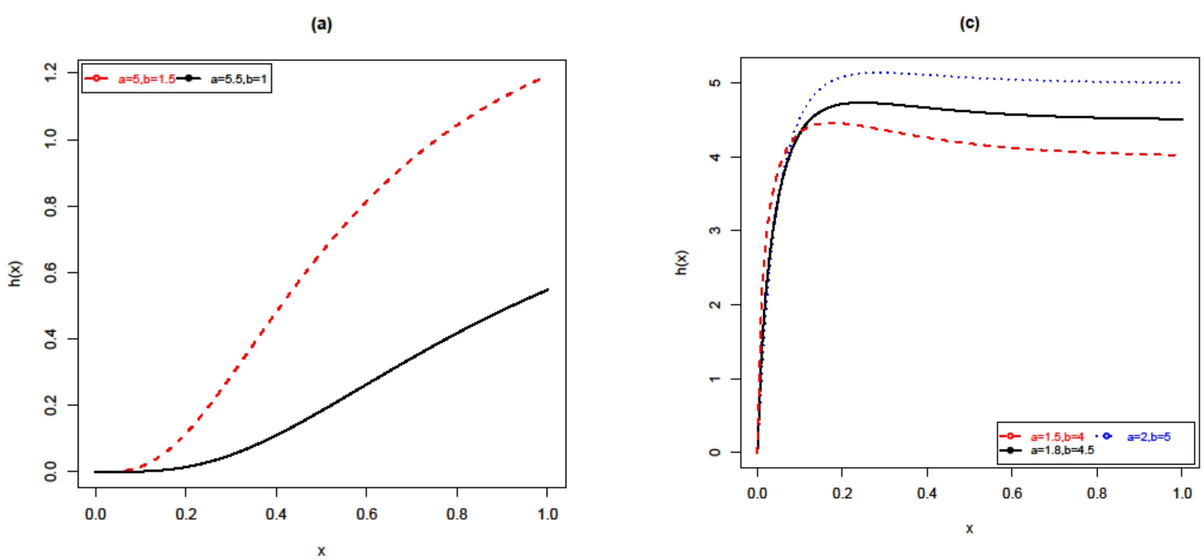
4. Moments and Generating Function

Now we study the moment and moment generating function of the PGKME distribution.

4.1 Moment

We can study some of the most important properties and characteristics of a distribution via moments, dispersion, skewness, kurtosis etc. So here we derive the r^{th} raw moment of the PGKME distribution.

$$E(Y^r) = \frac{\alpha\lambda e^{\alpha}}{(e-1)^{\alpha}}\int_0^{\infty} y^r e^{-(1+\lambda y - e^{-\lambda y})}\left[1 - e^{-(1-e^{-\lambda y})}\right]^{\alpha-1} dy, r \geq 1.$$



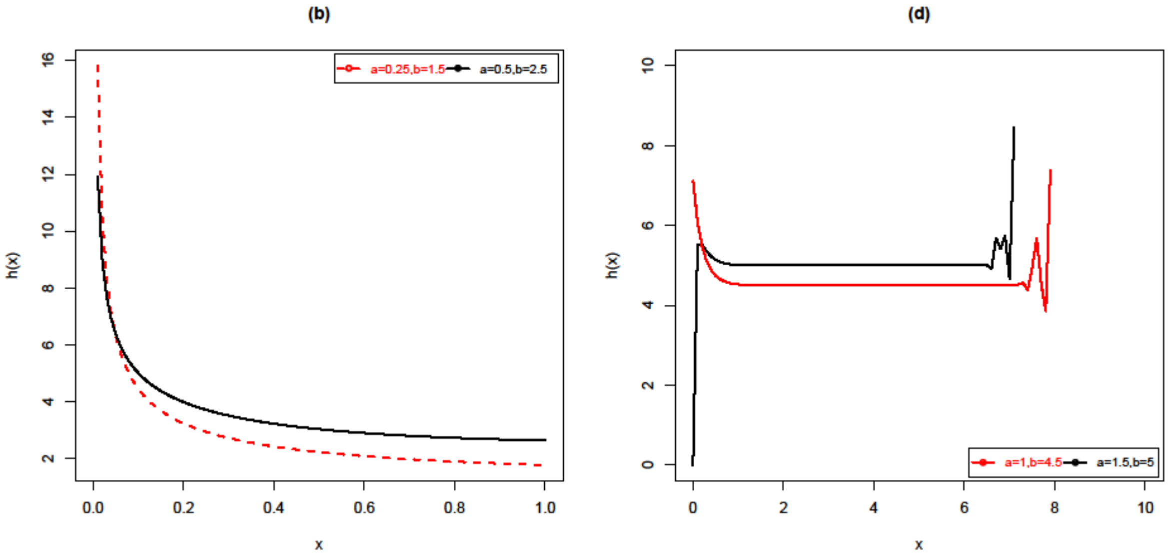


Fig 2: Failure rate function of PGKME (α, λ) for different parameter values of α and λ.

$$E(Y^r) = \frac{\alpha\lambda e^{\alpha}}{(e-1)^{\alpha}} \sum_{m=0}^{(\alpha-1)} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{(-1)^{m+j+k}}{j!} \binom{\alpha-1}{m} \binom{j}{k} \frac{\Gamma(r+1)}{\left(\lambda(k+1)\right)^{r+1}}.$$

4.2 Moment Generating Function

The mgf of the PGKME distribution is

$$M_Y(t) = \frac{\alpha\lambda e^{\alpha}}{(e-1)^{\alpha}} \sum_{m=0}^{(\alpha-1)} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{(-1)^{m+j+k}}{j!} \binom{\alpha-1}{m} \binom{j}{k} \frac{1}{(\lambda(k+1)-t)}.$$

5. Quantile Function of The Model

The p^{th} quantile function $Q(p)$ is obtained by the equation $F\big(Q(p)\big) = p$. The p^{th} quantile function of the PGKME distribution is

$$Q(p) = \frac{-1}{\lambda} log\left(1 + log\left(1 - p^{\frac{1}{\alpha}}\left(\frac{e-1}{e}\right)\right)\right). \tag{10}$$

Substituting $p = \frac{1}{4}, \frac{1}{2}$ and $\frac{3}{4}$ in above equation, we get the first, second and third quartiles respectively. A random sample of PGKME (α, λ) distribution can be simulated using

$$Y = \frac{-1}{\lambda} log\left(1 + log\left(1 - v_i^{\frac{1}{\alpha}}\left(\frac{e-1}{e}\right)\right)\right), \tag{11}$$

where $v_1, v_2, \ldots, v_n$ are independent random observations from the standard uniform distribution.

6. Mean Residual Life Function

The expected remaining life $(Z - z)$, given that the item has survived up to time z is the mean residual life (MRL). For the PGKME (α, λ) distribution

$$\eta(z) = E(Z - z | Z > z) = \frac{\int_0^{\infty} y g(y) dy}{1 - G(z)} - z$$

$$= \alpha\lambda e^{\alpha} \sum_{m=0}^{(\alpha-1)} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{(-1)^{m+j+k}}{j!} \binom{\alpha-1}{m} \binom{j}{k} \left(1 + \lambda z(k+1) e^{-\lambda z(k+1)}\right)$$

$$\left[((k+1)^{-2}\lambda^2)^1 - z(e-1)^\alpha - \left(e\left(1-e^{-\left(1-e^{-\lambda z}\right)}\right)\right)^\alpha\right]$$

7. Stochastic Order

Stochastic ordering of positive continuous random variables is an important tool for evaluating the comparative behaviour. A random variable Y is said to be smaller than a random variable Zin the

(i) stochastic order, $(Y \leq_{st} Z)$, if $G_Y(y) \geq G_Z(y)$for all y.

(ii) hazard rate order,$(Y \leq_{hr} Z)$, if $h_Y(y) \geq h_Z(y)$for all y.

(iii) mean residual life order, $(Y \leq_{mrl} Z)$, if $m_Y(y) \geq m_Z(y)$ for all y.

(iv) likelihood ratio order, $(Y \leq_{lr} Z)$, if $\frac{G_Y(y)}{G_Z(y)}$ decreases in y.

Shaked and Shanthikumar (Shaked &Shanthikumar (2007)) has been given for a detailed description about stochastic ordering and related results.

$$Y \leq_{lr} Z \Rightarrow Y \leq_{hr} Z \Rightarrow Y \leq_{mrl} Z$$
$$\Downarrow$$
$$Y \leq_{st} Z$$

The PGKME distribution is ordered with respect to the strongest 'likelihood ratio' ordering as shown in the following theorem:

Theorem 7.1

Let Y follows $PGKME(\alpha_1, \lambda_1)$ and Z follows $PGKME(\alpha_2, \lambda_2)$. If$\alpha_1 = \alpha_2$, and $\lambda_1 \geq \lambda_2$ (or if $\lambda_1 = \lambda_2$, and $\alpha_1 \geq \alpha_2$), then $Y \leq_{lr} Z$ and hence $X \leq_{hr} Z$ and $Y \leq_{st} Z$.

Proof.

$$\frac{g_Y(y)}{g_Z(y)} = \frac{e^{\alpha_1}(e-1)^{\alpha_2}\alpha_1\lambda_1 e^{-\left(1+\lambda_1 y - e^{-\lambda_1 y}\right)}\left(1-e^{-\left(1-e^{-\lambda_1 y}\right)}\right)^{\alpha_1 - 1}}{e^{\alpha_2}(e-1)^{\alpha_1}\alpha_2\lambda_2 e^{-\left(1+\lambda_2 y - e^{-\lambda_2 y}\right)}\left(1-e^{-\left(1-e^{-\lambda_2 y}\right)}\right)^{\alpha_2 - 1}}.$$

Thus

$$\frac{\partial}{\partial y} log \frac{g_Y(y)}{g_Z(y)} = \lambda_1\left(1+e^{-\lambda_1 y}\right) - (\alpha_1 - 1)\frac{e^{-\left(1+\lambda_1 y - e^{-\lambda_1 y}\right)}}{\left(1-e^{-\left(1-e^{-\lambda_1 y}\right)}\right)} - \lambda_2\left(1+e^{-\lambda_2 y}\right) - (\alpha_2 - 1)\frac{e^{-\left(1+\lambda_2 y - e^{-\lambda_2 y}\right)}}{\left(1-e^{-\left(1-e^{-\lambda_2 y}\right)}\right)}.$$

Case (i) if $\alpha_1 = \alpha_2 = \alpha$, and $\lambda_1 \geq \lambda_2$, then $\frac{\partial}{\partial y} log \frac{g_Y(y)}{g_Z(y)} < 0$. This means that $Y \leq_{lr} Z$ and hence $Y \leq_{hr} Z$ and $Y \leq_{st} Z$.

Case (ii) if $\alpha_1 \geq \alpha_2$, and $\lambda_1 = \lambda_2 = \lambda$, then $\frac{\partial}{\partial y} log \frac{g_Y(y)}{g_Z(y)} < 0$. This means that $Y \leq_{lr} Z$ and hence $Y \leq_{hr} Z$ and $Y \leq_{st} Z$.

8. Distribution of Maximum and Minimum

Let $Y_1, Y_2, \cdots, Y_n$ be a sample of size n from PGKME with cdf and pdf as in (6) and (5), respectively. Let $Y_{(1)}, Y_{(2)}, \cdots, Y_{(n)}$ denote the order statistics obtained from this sample.

$$g_{Y_{(r)}}(y;\alpha,\lambda) = \frac{n!}{(r-1)!\,(n-r)!} g(y;\alpha,\lambda) G^{r-1}(y;\alpha,\lambda)\overline{G}^{n-r}(y;\alpha,\lambda)$$

$$= \frac{n!}{(r-1)!(n-r)!}\frac{\alpha\lambda e^\alpha}{(e-1)^\alpha} e^{-\left(1+\lambda y - e^{-\lambda y}\right)}\left(1-e^{-\left(1-e^{-\lambda y}\right)}\right)^{\alpha-1}$$
$$\left(\frac{\left[e\left(1-e^{-\left(1-e^{-\lambda y}\right)}\right)\right]^\alpha}{(e-1)^\alpha}\right)^{r-1}\left(1-\frac{\left[e\left(1-e^{-\left(1-e^{-\lambda y}\right)}\right)\right]^\alpha}{(e-1)^\alpha}\right)^{n-r}.$$

The pdf of the n^{th} order statistics $Y_{(n)}$, is

$$g_{Y_{(n)}}(y;\alpha,\lambda) = \frac{\alpha\lambda e^\alpha}{(e-1)^\alpha} e^{-\left(1+\lambda y - e^{-\lambda y}\right)}\left(1-e^{-\left(1-e^{-\lambda y}\right)}\right)^{\alpha-1}$$

$$\left(\frac{\left[e\left(1-e^{-\left(1-e^{-\lambda y}\right)}\right)\right]^{\alpha}}{(e-1)^{\alpha}}\right)^{n-1}, \alpha>0, \lambda>0.$$

The cdf of $Y_{(n)}$ is

$$G_{Y_{(n)}}(y;\alpha,\lambda)=P\left(Y_{(n)}\leq y\right)=\left(\frac{\left[e\left(1-e^{-\left(1-e^{-\lambda y}\right)}\right)\right]^{\alpha}}{(e-1)^{\alpha}}\right)^{n}.$$

The pdf of the 1st order statistics $Y_{(1)}$, is

$$g_{Y_{(1)}}(y;\alpha,\lambda)=\frac{\alpha\lambda e^{\alpha}}{(e-1)^{\alpha}}e^{-\left(1+\lambda y-e^{-\lambda y}\right)}\left(1-e^{-\left(1-e^{-\lambda y}\right)}\right)^{\alpha-1}$$

$$\left(1-\frac{\left[e\left(1-e^{-\left(1-e^{-\lambda y}\right)}\right)\right]^{\alpha}}{(e-1)^{\alpha}}\right)^{n-1}.$$

The cdf of $Y_{(1)}$ is $G_{Y_{(1)}}(y;\alpha,\lambda)=P\left(Y_{(1)}\leq y\right)=1-\left(1-\frac{\left[e\left(1-e^{-\left(1-e^{-\lambda y}\right)}\right)\right]^{\alpha}}{(e-1)^{\alpha}}\right)^{n}.$

Reliability, $R(y;\alpha,\lambda)$, of series and parallel system having n components with independent and identically distributed (iid) PGKME(α,λ) distribution, respectively, are

$$\left(1-\frac{\left[e\left(1-e^{-\left(1-e^{-\lambda y}\right)}\right)\right]^{\alpha}}{(e-1)^{\alpha}}\right)^{n} \text{ and } 1-\left(\frac{\left[e\left(1-e^{-\left(1-e^{-\lambda y}\right)}\right)\right]^{\alpha}}{(e-1)^{\alpha}}\right)^{n}$$

9. Estimation of Parameters

In this section, we discuss some estimation methods for estimating the parameters of the PGKME (α,λ) distribution, namely, the maximum likelihood estimator, the moment estimator, and the least squares estimator.

9.1 Maximum Likelihood Estimation

The method of maximum likelihood estimation is the most popular method for finding estimates of the parameters of a distribution. Here we maximize the logarithm of the likelihood function to find the estimators. We use the maximum likelihood method to estimate the parameters of the PGKME distribution. Let $Y_1, Y_2, \cdots, Y_n$ be the random sample of size n from PGKME(α,λ) distribution. Then the likelihood function is defined as,

$$L(y;\alpha,\lambda)=\prod_{i=1}^{n}g(y_i;\alpha,\lambda).$$

In PGKME(α,λ) distribution,

$$L(y;\alpha,\lambda)=\frac{\alpha^n\lambda^n e^{n\alpha}}{(e-1)^{n\alpha}}e^{-\lambda\sum_{i=1}^{n}y_i}e^{-\sum_{i=1}^{n}\left(1-e^{-\lambda y_i}\right)}\prod_{i=1}^{n}\left(1-e^{-\left(1-e^{-\lambda y_i}\right)}\right)^{\alpha-1}$$

log-likelihood function of the distribution is given by

$$logL(y;\alpha,\lambda)=n\alpha log\left(\frac{e}{e-1}\right)+nlog\alpha+nlog\lambda-\lambda\sum_{i=1}^{n}y_i$$
$$+(\alpha-1)\sum_{i=1}^{n}log\left(1-e^{-\left(1-e^{-\lambda y_i}\right)}\right)-\sum_{i=1}^{n}\left(1-e^{-\lambda y_i}\right). \quad (12)$$

Partial derivatives of the log-likelihood function with respect to the parameters α and λ are

$$\frac{\partial logL}{\partial\alpha}=\frac{n}{\alpha}+nlog\left(\frac{e}{e-1}\right)+\sum_{i=1}^{n}log\left(1-e^{-\left(1-e^{-\lambda y_i}\right)}\right) \quad (13)$$

and

$$\frac{\partial logL}{\partial\lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}y_i-\sum_{i=1}^{n}\frac{(\alpha-1)y_i e^{-\left(1+\lambda y_i-e^{-\lambda y_i}\right)}}{\left(1-e^{-\left(1-e^{-\lambda y_i}\right)}\right)}-\sum_{i=1}^{n}y_i e^{-\lambda y_i}. \quad (14)$$

Equating Eqs. (13) and (14) to zero yields two non-linear equations. The maximum likelihood of the parameters is calculated α and λ obtained as the solution of these equations. Newton-Raphson method can be used to solve these equations with the help of R software.

Here we study the existence and uniqueness of the maximum likelihood estimates.

Theorem 9.1

Consider the right-hand side of the Eq. (13) and denoteas $d_1(\alpha;\lambda,y)$, where λ is the true value of the parameter. Then there exists at least one root for $d_1(\alpha;\lambda,y)=0$ for $\alpha\in(0,\infty)$ and the solution is unique.

Proof.

We have

$$d_1(\alpha;\lambda,y)=\frac{n}{\alpha}+nlog\left(\frac{e}{e-1}\right)+\sum_{i=1}^{n}log\left(1-e^{-\left(1-e^{-\lambda y_i}\right)}\right).$$

We get

$$\lim_{\alpha\to 0} d_1(\alpha;\lambda,y)=\infty,$$

and

$$\lim_{\alpha\to\infty} d_1(\alpha;\lambda,y)=-\infty<0.$$

Hence there exist at-least one root say $\hat{\alpha}\in(0,\infty)$ such that $d_1(\hat{\alpha};\lambda,y)=0$. The root is unique when the first derivative of $d_1(\hat{\alpha};\lambda,y)$, i.e., $d_1'(\hat{\alpha};\lambda,y)<0$, where

$$d_1'(\hat{\alpha};\lambda,y)=\frac{-n}{\alpha^2}. \quad \blacksquare$$

Theorem 9.2 *Consider the right-hand side of the Eq. (14) and denote as $d_2(\lambda;\alpha,y)$, where α is the true value of the parameter. Then there exists at least one root for $d_2(\lambda;\alpha,y)=0$ for $\lambda\in(0,\infty)$ and the solution is unique.*

Proof.

We have

$$d_2(\lambda;\alpha,y)=\frac{n}{\lambda}-\sum_{i=1}^{n}y_i-\sum_{i=1}^{n}\frac{(\alpha-1)y_ie^{-\left(1+\lambda y_i-e^{-\lambda y_i}\right)}}{\left(1-e^{-\left(1-e^{-\lambda y_i}\right)}\right)}-\sum_{i=1}^{n}y_ie^{-\lambda y_i}.$$

We get

$$\lim_{\lambda\to 0} d_2(\lambda;\alpha,y)=\infty,$$

and

$$\lim_{\lambda\to\infty} d_2(\lambda;\alpha,y)=-\sum y_i<0.$$

Hence there exist at-least one root say $\hat{\lambda}\in(0,\infty)$such that $d_2(\hat{\lambda};\alpha,y)=0$.The root is unique when the first derivative of $d_2(\hat{\lambda};\alpha,y)=0$, i.e., $d_2'(\hat{\lambda};\alpha,y)<0$, where

$$d_2'(\hat{\lambda};\alpha,y)=\frac{-n}{\lambda^2}-\sum_{i=1}^{n}{y_i}^2e^{-\lambda y_i}-(\alpha-1)\sum_{i=1}^{n}\frac{\left(y_i-y_ie^{-\lambda y_i}\right)e^{-\left(1+\lambda y_i-e^{-\lambda y_i}\right)}}{\left(1-e^{-\left(1-e^{-\lambda y_i}\right)}\right)}$$

$$-(\alpha-1)\sum_{i=1}^{n}\frac{y_ie^{-\left(2-2e^{-\lambda y_i}\right)}}{\left(1-e^{-\left(1-e^{-\lambda y_i}\right)}\right)^2}..$$

The joint probability for the location and scale parameters has a maximum point, and the maximum likelihood function is unimodal if the maximum is a stationary point.

9.1.1 Asymptotic Confidence Bounds

In this section, we compute the observed Fisher information for the MLE. Here we derive the asymptotic confidence intervals of the parameters involved in the PGKME(α,λ)distribution
when $\alpha>0$ and $\lambda>0$, by using variance covariance matrix. We now derive the observed Fisher information for the likelihood using (13) and (14). We have

$$I = E\begin{pmatrix} \frac{-\partial^2 logL}{\partial\alpha^2} & \frac{-\partial^2 logL}{\partial\alpha\partial\lambda} \\ \frac{-\partial^2 logL}{\partial\lambda\partial\alpha} & \frac{-\partial^2 logL}{\partial\lambda^2} \end{pmatrix}$$

where,

$$\frac{\partial^2 logL}{\partial\alpha^2} = \frac{-n}{\alpha^2},$$

$$\frac{\partial^2 logL}{\partial\lambda\partial\alpha} = \sum_{i=1}^{n} \frac{y_i e^{-\left(1+\lambda y_i - e^{-\lambda y_i}\right)}}{\left(1 - e^{-\left(1-e^{-\lambda y_i}\right)}\right)},$$

$$\frac{\partial^2 logL}{\partial\lambda^2} = \frac{-n}{\lambda^2} - \sum_{i=1}^{n} {y_i}^2 \, e^{-\lambda y_i}$$

$$-(\alpha - 1)\sum_{i=1}^{n} \frac{\left(y_i - y_i e^{-\lambda y_i}\right) e^{-\left(1+\lambda y_i - e^{-\lambda y_i}\right)}}{\left(1 - e^{-\left(1-e^{-\lambda y_i}\right)}\right)}$$

$$-(\alpha - 1)\sum_{i=1}^{n} \frac{y_i e^{-\left(2-2e^{-\lambda y_i}\right)}}{\left(1 - e^{-\left(1-e^{-\lambda y_i}\right)}\right)^2}.$$

We can derive the $(1-\delta)100\%$ confidence intervals of the parameters α and λ by using varaince-covariance matrix as in the form

$$\hat{\alpha} \pm Z_{\frac{\delta}{2}} \sqrt{Var(\hat{\alpha})}$$

and

$$\hat{\lambda} \pm Z_{\frac{\delta}{2}} \sqrt{Var(\hat{\lambda})}$$

where $Z_{\frac{\delta}{2}}$ is the upper $\left(\frac{\delta}{2}\right)^{\text{th}}$ percentile of the standard Normal distribution.

9.2 Method of Moment Estimation

The r^{th} moment of the PGKME(α, λ) distribution is

$$E(Y^r) = \frac{\alpha\lambda e^{\alpha}}{(e-1)^{\alpha}} \sum_{m=0}^{(\alpha-1)} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{(-1)^{m+j+k}}{j!} \binom{\alpha-1}{m} \binom{j}{k} \frac{\Gamma(r+1)}{\left(\lambda(k+1)\right)^{r+1}}.$$

Taking $r = 1,2$ in the above equation, we get the first and second raw moments as

$$\mu_1' = \frac{\alpha\lambda e^{\alpha}}{(e-1)^{\alpha}} \sum_{m=0}^{(\alpha-1)} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{(-1)^{m+j+k}}{j!} \binom{\alpha-1}{m} \binom{j}{k} \frac{1}{\left(\lambda(k+1)\right)^2}, \qquad (15)$$

and

$$\mu_2' = \frac{\alpha\lambda e^{\alpha}}{(e-1)^{\alpha}} \sum_{m=0}^{(\alpha-1)} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{(-1)^{m+j+k}}{j!} \binom{\alpha-1}{m} \binom{j}{k} \frac{2}{\left(\lambda(k+1)\right)^3}. \qquad (16)$$

The variance and coefficient of variation (CV) of the PGKME(α, λ) distribution is

$$Var(Y) = \frac{\alpha\lambda e^{\alpha}}{(e-1)^{\alpha}} \left[\frac{\alpha\lambda e^{\alpha}}{(e-1)^{\alpha}} \sum_{m=0}^{(\alpha-1)} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{(-1)^{m+j+k}}{j!} \binom{\alpha-1}{m} \binom{j}{k} \frac{2}{\left(\lambda(k+1)\right)^3} - \frac{\alpha\lambda e^{\alpha}}{(e-1)^{\alpha}} \left(\sum_{m=0}^{(\alpha-1)} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{(-1)^{m+j+k}}{j!} \binom{\alpha-1}{m} \binom{j}{k} \frac{1}{\left(\lambda(k+1)\right)^2} \right) \right]^2$$

and

$$CV = \sum_{m=0}^{(\alpha-1)}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^{m+j+k}}{j!}\binom{\alpha-1}{m}\binom{j}{k}\frac{2}{(\lambda(k+1))^3} - \frac{\alpha\lambda e^{\alpha}}{(e-1)^{\alpha}}\left(\sum_{m=0}^{(\alpha-1)}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^{m+j+k}}{j!}\binom{\alpha-1}{m}\binom{j}{k}\frac{1}{(\lambda(k+1))^2}\right)\left(\sum_{m=0}^{(\alpha-1)}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^{m+j+k}}{j!}\binom{\alpha-1}{m}\binom{j}{k}\frac{1}{(\lambda(k+1))^2}\right)^{-1}.$$

The method of moment estimators for the parameters of the PGKME(α,λ) distribution is obtained by equating Eqs. (15) and (16) with the sample moments respectively.

$$\frac{1}{n}\sum_{i=1}^{n} y_i = \frac{\hat{\alpha}\hat{\lambda}e^{\hat{\alpha}}}{(e-1)^{\hat{\alpha}}}\sum_{m=0}^{(\hat{\alpha}-1)}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^{m+j+k}}{j!}\binom{\hat{\alpha}-1}{m}\binom{j}{k}\frac{1}{\left(\hat{\lambda}(k+1)\right)^2},$$

and

$$\frac{1}{n}\sum_{i=1}^{n} {y_i}^2 = \frac{\hat{\alpha}\hat{\lambda}e^{\hat{\alpha}}}{(e-1)^{\hat{\alpha}}}\sum_{m=0}^{(\hat{\alpha}-1)}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^{m+j+k}}{j!}\binom{\hat{\alpha}-1}{m}\binom{j}{k}\frac{2}{\left(\hat{\lambda}(k+1)\right)^3}.$$

9.3 Method of Least-square Estimation

Let $Y_{(1)} < Y_{(2)} < \cdots < Y_{(n)}$ be the ordered random variables of a set of random samples$\{Y_1, Y_2, \cdots, Y_n\}$ of size n. $G\left(Y_{(i)}\right)$ is the distribution function of the i^{th} order statistic. The least squares estimators of the parameters α and λ are obtained by minimizing

$$\sum_{i=1}^{n}\left[G\left(Y_{(i)}\right) - \frac{i}{n+1}\right]^2 = \sum_{i=1}^{n}\left[\frac{e^{\alpha}}{(e-1)^{\alpha}}\left(1 - e^{-\left(1-e^{-\lambda y_{(i)}}\right)}\right)^{\alpha} - \frac{i}{n+1}\right]^2$$

with respect to α and λ.

10. Simulation

A simulation study is performed to verify that the MLEs perform for different sample sizes and different parameter values for the PGKME(α,λ) distribution using the Monte Carlo simulation method. In each experiment, using Eq. (11), 1000 pseudo-random samples have been generated for different values of parameters ($\alpha = 0.5,1,2$ and $\lambda = 0.75,1,1.5$) and sample sizes ($n = 25,50,75,100$). The standard error (SE), Bias and mean squared error (MSE) of the estimated parameters are computed. The results are obtained using R software and presented in Table 1, 2 and 3.

Table 1: The result of simulation study of PGKME distribution when λ = 0.75.

Parameter	N	MLE	SE	Bias	MSE	95% CI
$\alpha = 0.5$	25	$\hat{\alpha}$ = 0.5549	0.1198	5.488×10^{-5}	3.012×10^{-6}	(0.3578, 0.7519)
		$\hat{\lambda}$ = 0.8870	0.2612	1.370×10^{-4}	1.877×10^{-5}	(0.4574, 1.3166)
	50	$\hat{\alpha}$ = 0.5268	0.0854	2.682×10^{-5}	7.195×10^{-7}	(0.3864, 0.6672)
		$\hat{\lambda}$ = 0.8253	0.1921	7.532×10^{-5}	5.672×10^{-6}	(0.5094, 1.1413)
	75	$\hat{\alpha}$ = 0.5052	0.0693	1.517×10^{-5}	2.3×10^{-7}	(0.4011, 0.6292)
		$\hat{\lambda}$ = 0.7944	0.1569	4.444×10^{-5}	1.975×10^{-6}	(0.5362, 1.0526)
	100	$\hat{\alpha}$ = 0.5105	0.0603	1.053×10^{-5}	1.108×10^{-7}	(0.4114, 0.6097)
		$\hat{\lambda}$ = 0.7786	0.1362	2.865×10^{-5}	8.209×10^{-7}	(0.5547, 1.0026)

$\alpha = 1$	25	$\hat{\alpha} = 1.1426$	0.2652	1.426×10^{-4}	2.032×10^{-5}	(0.7063, 1.5788)
		$\hat{\lambda} = 0.8487$	0.2151	9.87×10^{-5}	9.751×10^{-6}	(0.4950, 1.2025)
	50	$\hat{\alpha} = 1.5074$	0.1878	5.744×10^{-5}	3.299×10^{-6}	(0.7484, 1.3664)
		$\hat{\lambda} = 0.7966$	0.1577	4.663×10^{-5}	2.174×10^{-6}	(0.5372, 1.0560)
	75	$\hat{\alpha} = 1.0381$	0.1555	3.815×10^{-5}	1.455×10^{-6}	(0.7824, 1.2939)
		$\hat{\lambda} = 0.7778$	0.1291	2.780×10^{-5}	7.726×10^{-7}	(0.5655, 0.9901)
	100	$\hat{\alpha} = 1.0278$	0.1346	2.785×10^{-5}	7.7755×10^{-7}	(0.8064, 1.2493)
		$\hat{\lambda} = 0.7699$	0.112	1.993×10^{-5}	3.973×10^{-7}	(0.5854, 0.9545)
$\alpha = 2$	25	$\hat{\alpha} = 2.3040$	0.5773	3.040×10^{-4}	9.243×10^{-5}	(1.3545, 3.2536)
		$\hat{\lambda} = 0.8089$	0.1805	5.888×10^{-5}	3.467×10^{-6}	(0.5120, 1.1057)
	50	$\hat{\alpha} = 2.1873$	0.4308	1.873×10^{-4}	3.506×10^{-5}	(1.4787, 2.8958)
		$\hat{\lambda} = 0.7962$	0.1346	4.619×10^{-5}	2.133×10^{-6}	(0.5748, 1.0176)
	75	$\hat{\alpha} = 2.1214$	0.3538	1.214×10^{-4}	1.475×10^{-5}	(1.5395, 2.7034)
		$\hat{\lambda} = 0.7807$	0.1109	3.074×10^{-5}	9.448×10^{-7}	(0.5983, 0.9632)
	100	$\hat{\alpha} = 2.0788$	0.3069	7.885×10^{-5}	6.217×10^{-6}	(1.5740, 2.5837)
		$\hat{\lambda} = 0.7645$	0.0959	1.448×10^{-5}	2.095×10^{-7}	(0.6068, 0.9221)

Table 2: The result of simulation study of PGKME distribution when λ = 1.

Parameter	N	MLE	SE	Bias	MSE	95% CI
$\alpha = 0.5$	25	$\hat{\alpha} = 0.5467$	0.1172	4.67×10^{-5}	2.185×10^{-6}	(0.3539, 0.7395)
		$\hat{\lambda} = 1.1796$	0.3456	1.796×10^{-4}	3.224×10^{-5}	(0.6111, 1.7481)
	50	$\hat{\alpha} = 0.5256$	0.0849	2.56×10^{-5}	6.55×10^{-7}	(0.3859, 0.6652)
		$\hat{\lambda} = 1.0855$	0.2538	8.552×10^{-5}	7.313×10^{-6}	(0.6680, 1.5029)
	75	$\hat{\alpha} = 0.5150$	0.06933	1.49×10^{-5}	2.23×10^{-7}	(0.4009, 0.6290)
		$\hat{\lambda} = 1.0612$	0.2103	6.117×10^{-5}	3.742×10^{-6}	(0.6437, 1.4787)
	100	$\hat{\alpha} = 0.512$	0.0627	1.201×10^{-5}	1.442×10^{-7}	(0.4129, 0.6111)
		$\hat{\lambda} = 1.0402$	0.1819	4.021×10^{-5}	1.617×10^{-7}	(0.7410, 1.3394)
$\alpha = 1$	25	$\hat{\alpha} = 1.1460$	0.2689	1.46×10^{-4}	2.132×10^{-5}	(0.7035, 1.5885)
		$\hat{\lambda} = 1.1303$	0.2875	1.303×10^{-4}	1.697×10^{-5}	(0.6574, 1.6031)
	50	$\hat{\alpha} = 1.0749$	0.1914	7.494×10^{-5}	5.617×10^{-6}	(0.7602, 1.3897)
		$\hat{\lambda} = 1.0759$	0.2113	7.593×10^{-5}	5.765×10^{-6}	(0.7283, 1.4235)
	75	$\hat{\alpha} = 1.0334$	0.1546	3.344×10^{-5}	1.118×10^{-6}	(0.7791, 1.2878)
		$\hat{\lambda} = 1.0388$	0.1729	3.876×10^{-5}	1.502×10^{-6}	(0.7544, 1.3231)
	100	$\hat{\alpha} = 1.0247$	0.1337	2.47×10^{-5}	6.101×10^{-7}	(0.8047, 1.2447)
		$\hat{\lambda} = 1.0135$	0.1478	1.353×10^{-5}	1.831×10^{-6}	(0.7704, 1.2567)

$\alpha = 2$	25	$\hat{\alpha}$ = 2.3467	0.5857	3.467×10^{-4}	1.202×10^{-4}	(1.3833, 3.3101)
		$\hat{\lambda}$ = 1.0135	0.2442	1.053×10^{-4}	1.108×10^{-5}	(0.7037, 1.5069)
	50	$\hat{\alpha}$ = 2.1578	0.4234	-0.001998	3.991×10^{-3}	(1.4613, 2.8543)
		$\hat{\lambda}$ = 1.0445	0.177	4.448×10^{-5}	1.978×10^{-6}	(0.7534, 2.8543)
	75	$\hat{\alpha}$ = 2.1126	0.3554	1.126×10^{-4}	1.268×10^{-5}	(1.5281, 2.6972)
		$\hat{\lambda}$ = 1.0329	0.1471	3.292×10^{-5}	1.084×10^{-6}	(0.7909, 1.2749)
	100	$\hat{\alpha}$ = 2.0505	0.3033	5.054×10^{-5}	2.554×10^{-6}	(1.5516, 2.5495)
		$\hat{\lambda}$ = 1.0153	0.1275	1.531×10^{-5}	2.345×10^{-7}	(0.8055, 1.2251)

Table 3: The result of simulation study of PGKME distribution when λ = 1.5.

Parameter	N	MLE	SE	Bias	MSE	95% CI
$\alpha = 0.5$	25	$\hat{\alpha}$ = 0.5553	0.1185	5.529×10^{-5}	3.057×10^{-6}	(0.3605, 0.7501)
		$\hat{\lambda}$ = 1.8463	0.5313	3.463×10^{-4}	1.2×10^{-4}	(0.9724, 2.7203)
	50	$\hat{\alpha}$ = 0.5281	0.0852	2.808×10^{-5}	7.883×10^{-7}	(0.3880, 0.6682)
		$\hat{\lambda}$ = 1.6372	0.3822	1.372×10^{-4}	1.882×10^{-5}	(1.009, 2.2659)
	75	$\hat{\alpha}$ = 0.5183	0.07	1.83×10^{-5}	3.347×10^{-7}	(0.4031, 0.6335)
		$\hat{\lambda}$ = 1.5780	0.3127	7.803×10^{-5}	6.089×10^{-6}	(1.0637, 2.0924)
	100	$\hat{\alpha}$ = 0.5122	0.0604	1.221×10^{-5}	1.489×10^{-7}	(0.4128, 0.6116)
		$\hat{\lambda}$ = 1.5609	0.2729	6.089×10^{-5}	3.707×10^{-6}	(1.1121, 2.0097)
$\alpha = 1$	25	$\hat{\alpha}$ = 1.1577	0.2708	1.577×10^{-4}	2.488×10^{-6}	(0.7123, 1.6031)
		$\hat{\lambda}$ = 1.7380	0.4418	2.379×10^{-4}	5.663×10^{-5}	(1.0112, 2.4647)
	50	$\hat{\alpha}$ = 1.0623	0.1892	6.233×10^{-5}	3.885×10^{-6}	(0.7512, 1.3734)
		$\hat{\lambda}$ = 1.6187	0.3187	1.187×10^{-4}	1.408×10^{-5}	(1.0944, 2.1429)
	75	$\hat{\alpha}$ = 1.0392	0.1558	3.925×10^{-5}	1.540×10^{-6}	(0.7829, 1.2955)
		$\hat{\lambda}$ = 1.5712	0.2614	7.120×10^{-5}	5.069×10^{-6}	(1.1412, 2.0012)
	100	$\hat{\alpha}$ = 1.0314	0.1355	3.144×10^{-5}	9.889×10^{-7}	(0.8086, 1.2542)
		$\hat{\lambda}$ = 1.5468	0.2255	4.681×10^{-5}	2.191×10^{-6}	(1.1759, 1.9177)
$\alpha = 2$	25	$\hat{\alpha}$ = 2.3418	0.5813	3.418×10^{-4}	3.991×10^{-3}	(1.3856, 3.2979)
		$\hat{\lambda}$ = 1.6497	0.3653	1.497×10^{-4}	2.24×10^{-5}	(1.0488, 2.2506)
	50	$\hat{\alpha}$ = 2.1807	0.431	1.807×10-4	3.264×10^{-5}	(1.4717, 2.8856)
		$\hat{\lambda}$ = 1.5867	0.2686	8.674×10^{-5}	7.525×10^{-6}	(1.1449, 2.0286)
	75	$\hat{\alpha}$ = 2.1127	0.3543	1.127×10^{-4}	1.269×10^{-5}	(1.530, 2.6954)
		$\hat{\lambda}$ = 1.5509	0.2206	5.093×10^{-5}	2.593×10^{-6}	(1.1881, 1.9138)
	100	$\hat{\alpha}$ = 2.0770	0.3069	7.697×10^{-5}	5.924×10^{-6}	(1.5722, 2.5818)

		$\hat{\lambda}$ = 1.5359	0.1922	3.589×10^{-5}	1.289×10^{-6}	(1.2198, 1.8520)

From Tables 1, 2 and 3, we can conclude that all the parameters show stability and the bias caused by the estimates is close to zero. Also, as the sample size increases, the absolute bias decreases. Thus, the estimates tend toward the true parameter values as the sample size increases.

11. Real Data Applications

In this section, we compare our PGKME(α, λ) distribution with some well-known distributions in the literature to demonstrate the superior performance of the PGKME distribution. The distributions used for comparison are the generalized DUS exponential (GDUSE) proposed by Maurya et al. (2017), the exponentiated exponential (EE) distribution proposed by Kundu (2004) and the Power Lindley (PL) distribution proposed by Ghitany et al. (2013).

We use AIC (Akaike information criterion), BIC (Bayesian information criterion), HQC (Hannan–Quinn Information Criterion), LL (log-likelihood) value, p-value and K–S (Kolmogorov-Smirnov) test value for the comparison of the data sets. The AIC, BIC, and HQC are defined as

$$AIC = 2k - 2logL,$$
$$BIC = klog(n) - 2logL,$$
$$HQIC = -2logL + 2klog\big(log(n)\big)$$

where n is the sample size, k is the number of parameters, and L is the maximum value of the likelihood function for the considered distribution. The distribution which gets minimum AIC, BIC, HQC and K–S test values and maximum LL and p-value is more suitable to the data set. In this section we analyzed the model using R language.

11.1 Data Set I

Here we consider the lifetimes of 30 electronic components taken from power-line voltage spikes during electric storms (Meeker & Escobar (1998)) presented in given below. The data has been previously used by Nadarajah et al. (2011) and Tahir et al. (2016a).

275, 13, 147, 23, 181, 30, 65, 10, 300, 173, 106, 300, 300, 212, 300, 300, 300, 2, 261, 293, 88, 247, 28, 143, 300, 23, 300, 80, 245, 266.

The comparison table of PGKME(α, λ)distribution with other four distributions are presented in Table 4. From Table 4, we can see that PGKME(α, λ) distribution has lowest AIC, BIC, HQIC and K-S test values and largest LL and p-value compared to other given distributions. The comparison plot is given in Figure 3.

Table 4: MLEs of parameters, KS Statistic, p-value, AIC, BIC and HQIC of the model for the Data SetI

Model	**ML Estimates**	**KS Statistic**	**p-value**	**-LL**	**AIC**	**BIC**	**HQIC**
PGKME	$\hat{\alpha}$ = 1.2588 $\hat{\lambda}$ = 0.00487	0.21412	0.1214	185.1121	366.2241	363.4217	365.3276
GDUS-E	$\hat{\alpha}$ = 1.1542 $\hat{\lambda}$ = 0.00616	0.31472	0.00525	215.1121	426.2241	423.4217	425.3276
EE	$\hat{\alpha}$ = 1.1542 $\hat{\lambda}$ = 0.00616	0.29149	0.01222	186.3387	368.6774	365.875	367.7809
PL	$\hat{\alpha}$ = 0.7846 $\hat{\lambda}$ = 0.0356	0.21766	0.1165	185.8192	367.6385	364.8361	366.742

11.2 Data Set II

The second data set gives the failure times of 20 Electric Bulbs, (Murthy et al. (2004)).

1.32, 12.37, 6.56, 5.05, 11.58, 10.56, 21.82, 3.60, 1.33, 12.62, 5.36, 7.71, 3.53, 19.61, 36.63, 0.39, 21.35, 7.22, 12.42, 8.92.

The comparison table of PGKME(α, λ)distribution with other four distributions are presented in Table 5.

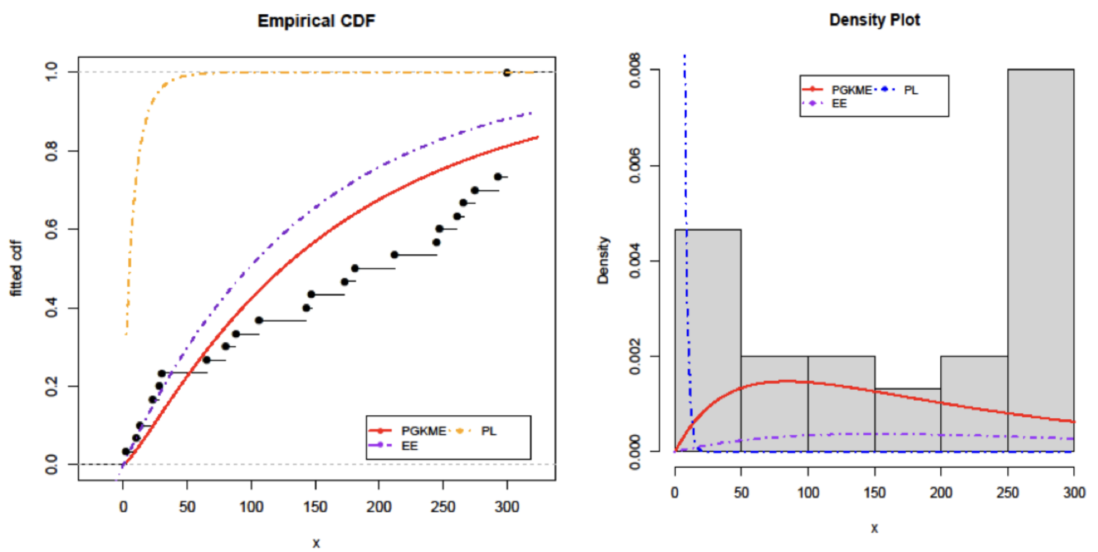


Fig 3: The Empirical cdf and Density Plot of the fitted distribution for the Data Set I.

From Table 5, we can see that PGKME(α, λ)distribution has lowest AIC, BIC, HQIC and K-S test values and largest LL and p-value compared to other given distributions except EE and PL distributions. When compared to EE and PL distributions AIC, BIC and HQIC values are slightly larger than PGKME(α, λ) distribution. Our model has lowest K-S test values and largest LL value and p-value compared to EE and PL distributions.

Table 5: MLEs of parameters, KS Statistic, p-value, AIC, BIC and HQIC of the model for the Data Set II

Model	**ML Estimates**	**KS Statistic**	**p-value**	**-LL**	**AIC**	**BIC**	**HQIC**
PGKME	$\hat{\alpha} = 1.5589$ $\hat{\lambda} = 0.0959$	0.09683	0.9826	67.02274	129.6174	127.626	129.2287
GDUS-E	$\hat{\alpha} = 1.3487$ $\hat{\lambda} = 0.1145$	0.66478	3.176×10^{-9}	86.5695	169.1389	167.1475	168.7502
EE	$\hat{\alpha} = 1.3486$ $\hat{\lambda} = 0.1145$	0.2914	0.05345	66.5695	129.1389	127.1475	128.7502
PL	$\hat{\alpha} = 0.9143$ $\hat{\lambda} = 0.2176$	0.1089	0.9511	66.53576	129.0715	127.0801	128.6828

The plot of empirical cdf and density of the distributions for the second data set is given in Figure 4.

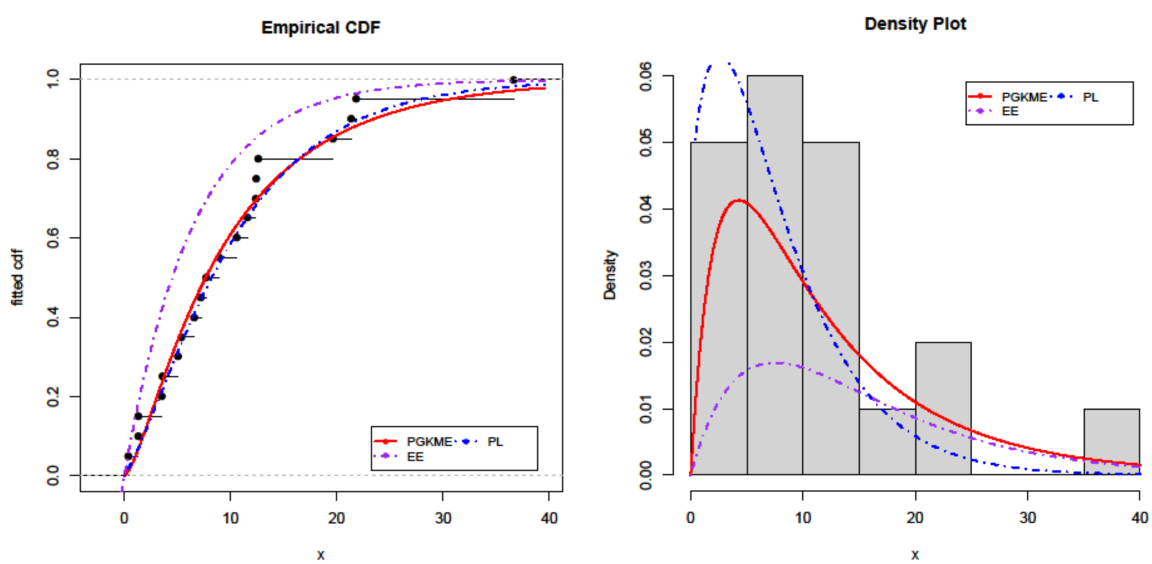


Fig 4: The Empirical cdf and density Plot of the fitted distribution for the second Data Set.

11.3 Data Set III

In this section, we apply our estimation methods in a real data set to verify the feasibility. This data set is taken from *Murthy et al. (2004)*. It shows 50 items put into use at $t = 0$ and failure items are recorded in weeks.

0.013, 0.065, 0.111, 0.111, 0.163, 0.309, 0.426, 0.535, 0.684, 0.747, 0.997, 1.284, 1.304, 1.647, 1.829, 2.336, 2.838, 3.269, 3.977, 3.981, 4.520, 4.789, 4.849, 5.202, 5.291, 5.349, 5.911, 6.018, 6.427, 6.456, 6.572, 7.023, 7.087, 7.291, 7.787, 8.596, 9.388, 10.261, 10.713, 11.658, 13.006, 13.388, 13.842, 17.152, 17.283, 19.418, 23.471, 24.777, 32.795, 48.105.

The comparison table of PGKME(α, λ)distribution with other four models are presented in Table 6. From, we can see that PGKME(α, λ)distribution has lowest AIC, BIC, HQIC and K-S test values and largest LL and p-value compared to other given distributions. The plot of Empirical cdf and density of the distributions for the second data set is given in Figure 5.

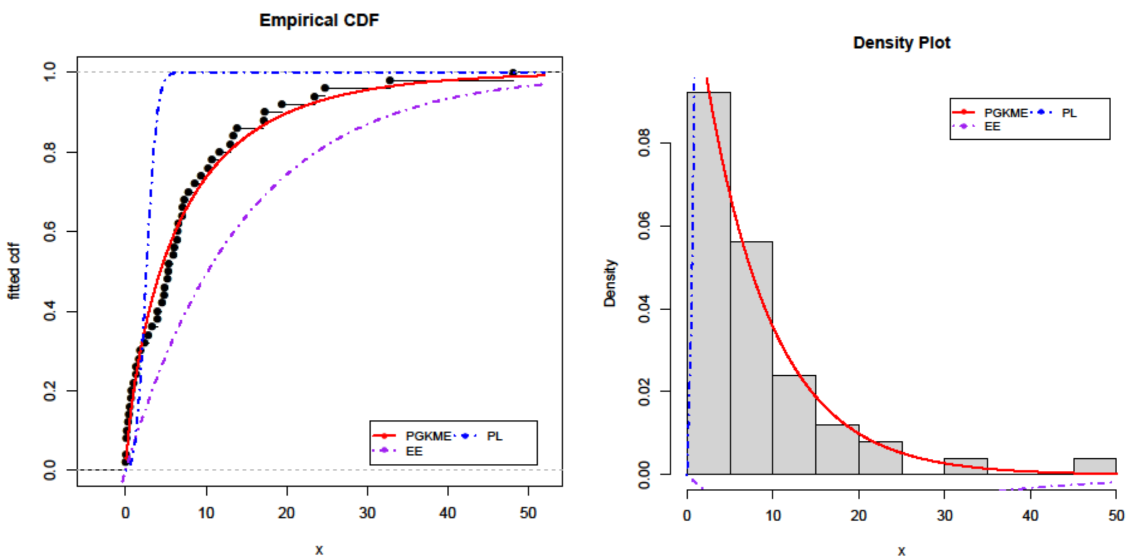


***Fig 5: The Empirical Cdf and Density Plot of the fitted distribution for the third Data Set*.**

Table 6: MLEs of parameters, KS Statistic, p-value, AIC, BIC and HQIC of the model for the Data SetIII

Model	**ML Estimates**	**KS Statistic**	**p-value**	**-LL**	**AIC**	**BIC**	**HQIC**
PGKME	$\hat{\alpha} = 0.7442$ $\hat{\lambda} = 0.0785$	0.1034	0.6589	150.6704	297.0361	293.212	295.5799
GDUS-E	$\hat{\alpha} = 0.6863$ $\hat{\lambda} = 0.0992$	0.4705	4.837×10^{-10}	200.2954	396.5909	392.7668	395.1347
EE	$\hat{\alpha} = 0.6863$ $\hat{\lambda} = 0.0992$	0.2888	0.000475	150.2954	296.5909	292.7668	295.1347
PL	$\hat{\alpha} = 0.6641$ $\hat{\lambda} = 0.4884$	0.119	0.4784	150.518	297.3407	293.5167	295.8845

12. Conclusion

In this work, we proposed a new lifetime model with increasing, decreasing and bathtub shaped hazard rate function. The distribution is obtained by Power generalized KM transformation. The basic properties of the new distribution, PGKME(α, λ) distribution, like moments, mgf, mean residual life function and quantile function are derived. We examined the Stochastic ordering. The consistency of the maximum likelihood estimators of the parameters involved in the distribution is proved through simulation studies. With the help of three real data sets, we studied and illustrated the flexibility of the new distribution.

The PGKME distribution is a promising model for large variety of lifetime data in the medical and reliability fields. So, we can say that PGKME distribution performs better than Generalized DUS Exponential, Exponentiated Exponential and Power Lindley distributions.

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