SCOPUA J. Appl. Stat. Res.
Vol.1, Issue 2, May 2025
DOI: 10.64060/JAS-R.v1.i2.0

A new generalized Logistic class of distributions: Properties and applications on flood and earthquake data sets with bivariate extension

Sadaf Khan[*1] **Farrukh Jamal**[2] **Muhammad Hussain Tahir**[1] **Ahmed Elhassanein**[3]

[1]Department of Statistics, The Islamia University of Bahawalpur, Bahawalpur, Pakistan
[2]Department of Statistics , Faculty of Science, University of Tabuk, Tabuk , Saudi Arabia
[3]Department of Mathematics, College of Science, University of Bisha, P.O. Box 551, Bisha 61922, Saudi Arabia
[*]Corresponding Author: smkhan6022@gmail.com

Received: 18 April 2025 Revised: 01 July 2025 Accepted: 22 July 2025 Published online: 16 August 2025

Abstract

For univariate and bi-variate data, we propose a new generalized logistic class of distributions flexible enough to exhibit monotone and non-monotone hazard rates shapes. The physical interpretation of the new family preludes in the context of series-parallel structures. Its mathematical features, including a valuable expansion for the density, explicit formulations for the quantile function, ordinary and incomplete moments, and generating function, are all derived. The parameter estimation of the new family is done using the maximum likelihood method. One of the unique model, called the generalized logistic Burr-III, is thoroughly investigated in applied sense. The flexible density and hazard rate shapes capacitates the model to be applicable in extreme value theory. For univariate case, two real-life data sets related to hydrology and seismic activity have been employed to solidify the superiority of the proposed distribution to five well established families. For bivariate data, initially a bivariate extension of the proposed family is established analytically with the help of empirical findings. Then, a real bi-variate data related to operational lifespan of two components of a computer, has been studied using bivariate generalized logistic Burr-III model and the results are reported.

Keywords: Generalized family; Logistic-G family; Burr-III distribution; Maximum likelihood; Bi-variate modeling; Surface plots

1 INTRODUCTION

In recent decades, new distributions have been proven a very effective tool for modeling real-life phenomenons and data generated in different fields. According to modern philosophy of *distribution theory*, the major reasons for proposing new models with acronyms *extended, modified or generalized* are the induction of shape parameter(s) in a known distribution or G-class, and/or defining effective logical special functions. Surely, one cannot ignore the use of transformation, compounding and convolution as per the required situation(s). Under these considerations, a practitioner or researcher can utilize flexible models that have ability to cope different data sets generated in many fields such as engineering, medical science, actuaries, agriculture, computing sciences, education, among others. The flexibility may be considered as: (i) flexible shapes of the density and hazard rates, (ii) attractive and rich mathematical properties, (iii) performance of estimators through simulation study, and (iv) numerical and graphical goodness-of-fit tests results, etc. The philosophy of inducting one shape-parameter either as power or tilt was pioneered by Azzalini [1] (skewed-G) and further extended by Marshal and Olkin [2] (Marshall-Olkin-G), Gupta et al. [3] (exponentiated-G), Gleaton and Lynch [4] (odd log-logistic-G), Shaw and Buckley [5] (transmuted-G), Zografos and Balakrishnan [6] (gamma-G type 1), Ristić and Balakrishnan [7] (gamma-G type 2), Bourguignon *et al.* [8] (odd Weibull-G), among many others. This parameter induction philosophy was extended to two and three shape parameters by Eugene *et al.* [9] (beta-G), Cordeiro and de-Castro [10] (Kumaraswamy-G), Cordeiro *et al.* [11] (exponentiated generalized-G), Alexander *et al.* [12] (McDonald-G), among a long list of others. Some of the recent literature related to generalized classes and their applications are provided in [13]-[22].The defining criteria plus terms and conditions for an effective generalized (G) class of distributions was pioneered by Alzaatreh *et al.* [23] which most of the G-classes follow. This generalized type was termed as transformed-transformer (T-X) approach that aims to define the probability density function (pdf) of a new G-class in terms of the pdf and cumulative distribution function (cdf) of a baseline model. The T-X framework generates different special flexible models fulfilling some conditions with support required in this regard. Suppose that $r(t)$ is the pdf and $R(t)$ is the cdf of a random variable (rv) $T \in [a, b]$, where the condition $-\infty < a < b < \infty$ holds, and $W[G(x)]$ is a generator function of the cdf $G(x)$ or survival function (sf) $\bar{G}(x) = 1 - G(x)$ of any baseline rv. Then, the generator must fulfill the following requirements:

(i) $W[G(x)] \in [a, b]$,
(ii) $W[G(x)]$ is differentiable and monotonically non-decreasing, and
(iii) $\lim_{x\to-\infty} W[G(x)] = a$ and $\lim_{x\to\infty} W[G(x)] = b$.

The cdf of the T–X family has the form

$$F_{TX}(x) = \int_a^{W[G(x)]} r(t)\,dt = R\Big(W\big[G(x)\big]\Big), \tag{1}$$

where $W[G(x)]$ satisfies the conditions (i)–(iii).

The pdf corresponding to Equation (1) is

$$f_{TX}(x) = r\left(W\big[G(x)\big]\right)\frac{d}{dx}W\big[G(x)\big]. \tag{2}$$

Let $T \in (-\infty, \infty)$ be a standard logistic rv with density

$$r(t) = \mathrm{e}^{-t}\,(1 + \mathrm{e}^{-t})^{-2}, \quad t \in \mathbb{R}. \tag{3}$$

A few G-classes defined from the logistic distribution have been reported in literature such as those listed in Table 1.

The main aim of this article is to establish the foundation of a new extended logistic family having a flexible link function $W[G(x)] = \log\left[-\log\left\{1 - \left[1 - \bar{G}(x)^{\alpha}\right]^{\beta}\right\}\right]$ with two additional shape parameters.

Table 1: Logistic-G classes reported in literature.

SNo.	Generator $W[G(x)]$ used	Authors	Name of the class
1	$\log\left[G(x)/\bar{G}(x)\right]$	Torabi and Montazari [24]	Logistic-G of type 1
2	$\log\left[-\log\,\bar{G}(x)\right]$	Tahir et al. [25]	Logistic-X of type 2
3	$\log\left[-\log\,G(x)\right]$	Mansoor et al. [26]	Logistic-G of type 3

For convenience, henceforth, let $w = \left[1-\bar{G}(x)^{\alpha}\right]^{\beta}$. For brevity, the proposed generator is then represented by the notation $W[G(x)] = \log[-\log(1-w)]$ throughout the manuscript.

The prime motivations to introduce new logistic class include: (a) the distinctive link function allows the applicability of the proposed family in physical processes with series-parallel structures; (b) Analytically speaking, the two additional parameters in the distinctive link function play a significant role in addressing the identifiability issues of inverted models as compared to existing Logistic-G families;(c) the added flexibility due to $W[G(x)]$ is visible in studying the special model (see section 3) with the ability to accommodate a wide range of density and hazard rate shapes in comparison to special models studied in the existing Logistic-G classes; (d) the suggested extension is capable of generating skewed distributions with substantially larger tails, a scenario which usually occurs in extreme value theory; (e) in the course of literature review, it was observed that the initiated family of distributions serve as a parent model of several existing sub-families (see Table 12); (f) the uniqueness of this study can be stated with the mere fact that the current literature lacks the bi-variate extension of any of already proposed Logistic-G families (mentioned in Table-1 of the manuscript). Hopefully, this article will fill this void in contemporary literature in which a bivariate extension of the proposed family is established analytically with the help of empirical findings of real data. We organize the article as follows. In Section 2, we define a new *generalized logistic* (GL) class, obtain a useful expansion for its density, explicit expressions for the moments and quantile and generating functions, and estimate the parameters. In Section 3, the Burr III (BIII) distribution is considered for baseline to propose a special model called the *generalized logistic-Burr-III* (GLBIII) distribution, whose mathematical properties along with empirical illustrations are investigated. Section 4 comprises of the bi-variate extension of GLBIII model with examples and real life application. Some concluding remarks are addressed in Section 5.

2 THE NEW GENERALIZED LOGISTIC CLASS

The cdf of the GL class follows from the T-X framework and the logistic cdf as

$$F(x;\alpha,\beta,\xi) = \int_{-\infty}^{\log[-\log(1-w)]} \mathrm{e}^{-t}\,\left(1+\mathrm{e}^{-t}\right)^{-2} dt = \left\{1+\left[-\log(1-w)\right]^{-1}\right\}^{-1}, \quad x \in \mathbb{R}, \tag{4}$$

where $\alpha > 0$ and $\beta > 0$ are shape parameters and ξ is the vector of baseline parameters. The pdf corresponding to (4) can be expressed as

$$f(x;\alpha,\beta,\xi) = \frac{\alpha\,\beta\,w g(x;\xi)\bar{G}^{\alpha-1}(x;\xi)}{w^{1/\beta}(1-w)\left[-\log\left(1-w\right)\right]^{2}}\left\{1+\left[-\log(1-w)\right]^{-1}\right\}^{-2}, \tag{5}$$

where $G(x;\xi)$, $g(x;\xi)$ and $\bar{G}(x;\xi)$ are the baseline cdf, pdf and sf. Further, omitting dependence on the vector ξ of parameters, we write $G(x) = G(x;\xi)$ and $g(x) = g(x;\xi)$.Henceforth, let $X \sim \mathrm{GL}(\alpha,\beta,\xi)$ be a rv having density (5). The hazard rate function (hrf) of X follows as

$$\begin{aligned}\tau(x;\alpha,\beta,\xi) \quad &= \quad \frac{\alpha\,\beta\,w\,g(x;\xi)\bar{G}^{\alpha-1}(x;\xi)}{w^{1/\beta}(1-w)\left[-\log\left(1-w\right)\right]^{2}}\left\{1+\left[-\log(1-w)\right]^{-1}\right\}^{-2} \\ &\quad \times\left[1-\left\{1+\left[-\log(1-w)\right]^{-1}\right\}^{-1}\right]^{-1}. \end{aligned} \tag{6}$$

The quantile function (qf) of X is determined by inverting (4)

$$Q(u) = Q_G\left[1 - \left\{1 - \left[1 - \mathrm{e}^{-\left(\frac{u}{1-u}\right)}\right]^{1/\beta}\right\}^{1/\alpha}; \xi\right], \; 0 < u < 1, \tag{7}$$

where $Q_G(\cdot) = Q_G(\cdot;\xi) = G^{-1}(\cdot;\xi)$ is the baseline qf.Equation (7) is very simple to obtain the skewness and kurtosis of X based on quantiles. The unique $W[G(x)]$ is superior to some of the existing G classes defined in current literature. Some special sib-classes arising from the proposed GL class are listed in Table 2.

Table 2: The GL class in relation to sub-classes in literature.

S No.	α	β	log	Range	Reduced distribution
1	α	β	✓	$(-\infty, +\infty)$	GL family (proposed)
2	1	1	✓	$(-\infty, +\infty)$	Logistic-X (Tahir et al. [25])
3	α	β	-	$(0, +\infty)$	Exponentited-generalized T-X (Nasiru et al. [27])
4	α	1	-	$(0, +\infty)$	Exponentiated T-X (Alzaghal et al. [28])
5	1	1	-	$(0, +\infty)$	T-X (Alzatreh et al. [23])

2.1 Analytic shapes of the density and hazard rate functions

The shapes of the density and hrf of X can be described analytically. The critical points of the density of X are the roots of the equation:

$$\frac{g(x)g'(x)}{(1-w^{1/\beta})^{1/\alpha}}\left[\frac{(1-\alpha)w^{1/\beta} - \alpha(\beta-1)(1-w^{1/\beta})}{w^{1/\beta}}\right] + \frac{2\alpha\,\beta\, g(x)g'(x)(1-w^{\frac{1}{\beta}})^{\frac{\alpha-1}{\alpha}}}{-\log(1-w)}\left[\frac{1}{(1-w)} - \frac{(1-w)w^{\frac{\beta-1}{\beta}}}{-\log(1-w)\{1+[-\log(1-w)]^{-1}\}}\right] = 0\,. \tag{8}$$

The critical points of the hrf of X are obtained from the equation

$$\frac{g'(x)}{g(x)} + \frac{g(x)g'(x)}{(1-w^{1/\beta})^{1/\alpha}}\left[\frac{(1-\alpha)w^{1/\beta} - \alpha(\beta-1)(1-w^{1/\beta})}{w^{1/\beta}}\right] + \frac{\alpha\,\beta\, g(x)g'(x)(1-w^{\frac{1}{\beta}})^{\frac{\alpha-1}{\alpha}}w^{\frac{\beta-1}{\beta}}}{(1-w)}\left[1+\frac{2}{\log(1-w)}\right] + \frac{\alpha\,\beta\, g(x)g'(x)(1-w^{\frac{1}{\beta}})^{\frac{\alpha-1}{\alpha}}w^{\frac{\beta-1}{\beta}}[-\log(1-w)]^{-2}}{(1-w)\left\{1+[-\log(1-w)]^{-1}\right\}} \times\left[\frac{\left\{1+[-\log(1-w)]^{-1}\right\}}{1-\left\{1+[-\log(1-w)]^{-1}\right\}^{-1}}\right] = 0\,. \tag{9}$$

Equations (8) and (9) are complex and we recommend to use any numerical software to determine the local maximum and minimum and the points of inflexion.

2.2 Useful expansion

We provide a useful linear combination for (5) in terms of exponentiated-G (exp-G) densities. Given a baseline cdf $G(x)$, a rv is said to have the exp-G distribution with power parameter $l > 0$, say Y $\sim$exp-G(l), if its cdf and pdf are

$$H_l(x) = G(x)^l \qquad \text{and} \qquad h_l(x) = l\, G^{l-1}(x)\, g(x),$$

respectively. For detailed discussion on exp-G distributions, we refer to Mudholkar and Srivastava [29], Gupta and Kundu [30], Nadarajah and Kotz [31], to just name a few.Consider two expansions that shall produce fruitful results throughout the manuscript. For any real parameter c and $z \in (0,1)$, the power series holds.

$$[-\log(1-z)]^c = \sum_{m=0}^{\infty} P_m(c)\, z^{(m+1)c}\,, \tag{10}$$

where $P_0(c) = 1/2$, $P_1(c) = c(3c+5)/24$, $P_2(c) = c(c^2+5c+6)/48$, etc, are the Stirling's polynomials. The reciprocal of a power series $\sum_{n=0}^{\infty} f_n\, z^n$ (given that $f_0 \neq 0$) exists as follows (Apostol [32])

$$\frac{1}{\sum_{n=0}^{\infty} f_n\, z^n} = \frac{1}{f_0} \sum_{n=0}^{\infty} g_n\, z^n\,, \tag{11}$$

where $g_0 = 1$ and $g_n = (-1)\sum_{i=1}^{n} f_i\, g_{n-i}$ (for $n \geq 1$).
Expanding (4) by the power series (10), we obtain

$$F(x) = \left[1 + \sum_{m=0}^{\infty} P_m(-1)\left(1-\bar{G}^{\alpha}\right)^{-(m+1)\beta}\right]^{-1}. \tag{12}$$

By applying the binomial expansion in Equation (12)

$$F(x) = \left[1 + \sum_{l=0}^{\infty} \omega_l\, G(x)^l\right]^{-1}\,,$$

where $\omega_l = \sum_{m,j=0}^{\infty} (-1)^j\, P_m(-1)\binom{-(m+1)\beta}{j}\binom{j\,\alpha}{l}$ (for $l \geq 0$).
Further, we can rewrite $F(x)$ as

$$F(x) = \left[\sum_{l=0}^{\infty} v_l\, G(x)^l\right]^{-1}\,,$$

where $v_0 = 1 + \omega_0$ and $v_l = \omega_l$ (for $l \geq 1$).
By using the result (11) in the last expression,

$$F(x) = \frac{1}{v_0} \sum_{l=0}^{\infty} d_l\, G^l(x)\,, \tag{13}$$

where $d_0 = 1$, and $d_l = (-1)\sum_{i=1}^{l} v_l\, d_{l-i}$ (for $l \geq 1$).
By simple differentiating of $F(x)$, we obtain

$$f(x) = \frac{1}{v_0} \sum_{l=0}^{\infty} d_{l+1}\, h_{l+1}(x;\xi)\,, \tag{14}$$

where $h_{l+1}(x;\xi) = (l+1)\, G^l(x;\xi)\, g(x;\xi)$ is the exp-G density with power parameter $l+1$. Equation (14) reveals that the GL density function is a linear combination of exp-G densities.

2.3 Mathematical properties

The formulae derived throughout the paper can be easily handled in most symbolic computation platforms (Maple, Mathematica and Matlab) which have currently the ability to deal with analytic expressions of formidable size and complexity. Henceforth, let Y_{l+1} be a rv having the exp-G distribution with power parameter $l+1$. We can find some mathematical quantities of the GL family from (14) and those ones of the exp-G distribution, which are known for at least fifty distributions; see those distributions listed in Tables 1, 3 and 5 of Tahir and Nadarajah [13].Firstly, the nth ordinary moment of X follows from (14) as

$$\mu'_n = \frac{1}{v_0} \sum_{l=0}^{\infty} d_{l+1}\, \mathbb{E}(Y_{l+1}^n) = \frac{1}{v_0} \sum_{l=0}^{\infty} (l+1)\, d_{l+1}\, \tau_{n,l}, \tag{15}$$

where $\mathbb{E}(Y_{l+1}^n) = \int_0^\infty y^n\, h_{l+1}(y;\xi)dy$ and $\tau_{n,l} = \int_0^1 Q_G(u)^n\, u^l du$. Setting $n = 1$ in (15) provides expressions for the means of several special distributions.Clearly, the central moments and cumulants of X can be determined from (15) using well-known relationships.Secondly, the nth lower incomplete moment of X, say $m_n(y) = \int_{-\infty}^y x^n\, f(x)dx$, is

$$m_n(y) \quad = \quad \frac{1}{v_0}\sum_{l=0}^{\infty} d_{l+1} \int_{-\infty}^{y} x^n\, h_{l+1}(x;\xi)dx = \frac{1}{v_0}\sum_{l=0}^{\infty} d_{l+1} \int_{0}^{G(y;\xi))} Q_G(u;\xi)^n\, u^l du. \tag{16}$$

The last two integrals can be evaluated numerically for most G distributions.Thirdly, for a given probability π, the Bonferroni and Lorenz curves (popular measures in economics, reliability, demography, insurance and medicine) of X are given by $B(\pi) = m_1(q)/(\pi\,\mu_1')$ and $L(\pi) = m_1(q)/\mu_1'$, respectively, where $q = Q(\pi)$ can be found from (7).Fourthly, the total deviations from the mean and median are $\delta_1 = 2\mu_1'\, F(\mu_1') - 2m_1(\mu_1')$ and $\delta_2 = \mu_1' - 2m_1(M)$, where $M = Q(0.5)$ and $F(\mu_1')$ comes from (4).Fifthly, the moment generating function (mgf) $M(t) = \mathbb{E}(\mathrm{e}^{t\,X})$ of X follows from (14) as

$$M(t) \quad = \quad \frac{1}{v_0}\sum_{l=0}^{\infty} d_{l+1}\, M_{l+1}(t) = \frac{1}{v_0}\sum_{l=0}^{\infty} (l+1)\, d_{l+1}\, \rho_l(t;\xi), \tag{17}$$

where $M_{l+1}(t)$ is the mgf of Y_{l+1} and $\rho_l(t;\xi) = \int_0^1 \exp[t\, Q_G(u;\xi)]\, u^l du$. Hence, we can obtain the mgfs of many special GL distributions directly from (17) and exp-G generating functions.

2.4 Estimation

We consider the estimation of the unknown parameters of the new class by the maximum likelihood method. Let $x_1, \cdots, x_n$ be n observations from the GL density class (5) with parameter vector $\boldsymbol{\Theta} = (\alpha, \beta, \xi^\top)^\top$. The log-likelihood $\ell = \ell(\boldsymbol{\Theta})$ for $\boldsymbol{\Theta}$ has the form

$$\begin{aligned} \ell \quad = & \quad n\log(\alpha) + n\log(\beta) + (\alpha-1)\sum_{i=1}^{n}\log\left(1-w^{1/\beta}\right)^{1/\alpha} + (\beta-1)\sum_{i=1}^{n}\log(w^{1/\beta}) \\ - & \quad \sum_{i=1}^{n}\log(1-w) - 2\sum_{i=1}^{n}\log[-\log(1-w)] - 2\sum_{i=1}^{n}\log\left[1+\{-\log(1-w)\}^{-1}\right]. \end{aligned} \tag{18}$$

The MLE $\widehat{\boldsymbol{\Theta}}$ of $\boldsymbol{\Theta}$ can be evaluated by maximizing (18). There are several routines for numerical maximization of $\ell(\theta)$ in the `R` program (`optim` function), `SAS` (`PROC NLMIXED`), `Ox` (sub-routine `MaxBFGS`), among others.Alternatively, we can differentiate the log-likelihood and solving the resulting nonlinear likelihood equations. Then, the score components with respect to α, β and ξ are

$$\begin{aligned} U_\alpha \quad = & \quad \frac{n}{\alpha} + \alpha\sum_{i=1}^{n}\log(1-w^{1/\beta})\, w_\alpha'(x_i;\alpha,\xi) - \sum_{i=1}^{n}\left(1-w^{1/\beta}\right) - 2\sum_{i=1}^{n}\frac{w_\alpha'(x_i;\xi)}{(1-w)[-\log(1-w)]} \\ & + (\beta-1)\sum_{i=1}^{n}\frac{w_\alpha'(x_i;\xi)}{w^{1/\beta}} + \sum_{i=1}^{n}\frac{w_\alpha'(x_i;\xi)}{1-w} + 2\sum_{i=1}^{n}\frac{(1-w)w_\alpha'(x_i;\xi)}{[-\log(1-w)]^2\,[1+\{-\log(1-w)\}^{-1}]}, \end{aligned}$$

$$\begin{aligned} U_\beta \quad = & \quad \frac{n}{\beta} + \left(\frac{\alpha-1}{\alpha}\right)\sum_{i=1}^{n}\frac{w^{1/\beta}\log(w)w_\beta'(x_i;\xi)}{\left(1-w^{1/\beta}\right)^{1/\alpha}} + \beta\sum_{i=1}^{n}\left[\log(w)\, w_\beta'(x_i;\xi)\right] + \sum_{i=1}^{n}\frac{w_\beta'(x_i;\xi)}{(1-w)} \\ & - \sum_{i=1}^{n}\log(w^{1/\beta}) - 2\sum_{i=1}^{n}\frac{w_\beta'(x_i;\xi)}{(1-w)[-\log(1-w)]}, \end{aligned}$$

$$U_{\xi_k} = \left(\frac{1-\alpha}{\alpha\,\beta}\right)\sum_{i=1}^{n}\left[\frac{w^{(1/\beta)-1}\,w'_{\xi}(x_i;\xi)}{(1-w^{1/\beta})}\right]+\left(\frac{1-\beta}{\beta}\right)\sum_{i=1}^{n}\frac{w'_{\xi}(x_i;\xi)}{w}+\sum_{i=1}^{n}\frac{w'_{\xi}(x_i;\xi)}{(1-w)}$$
$$-2\sum_{i=1}^{n}\frac{w'_{\xi}(x_i;\xi)}{[-\log(1-w)](1-w)}+2\sum_{i=1}^{n}\frac{(1-w)^{-1}w'_{\xi}(x_i;\xi)}{[1+\{-\log(1-w)\}^{-1}]\,[-\log(1-w)]^2},$$

where $w'_\alpha(x_i;\xi)$ and $w'_\beta(x_i;\xi)$ are, respectively, the derivatives with respect to α and β.

Setting the score components to zero and solving them simultaneously yields the MLEs of the parameters. The resulting equations cannot be solved analytically, but some statistical softwares can be used to solve them numerically through iterative Newton-Raphson type algorithms.

3 THE GL-BIII DISTRIBUTION

The BIII distribution has received much broader attention in distinct areas of modern science including income, wage or wealth size (by the name of Dagum distribution), in actuarial science (by the name of inverse Burr) and in meteorological science (by the name of Kappa distribution). According to Kleiber and Kotz [33], it is a special case of the four-parameter generalized beta II distribution. By taking the BIII model as baseline distribution with cdf $G(x;c,k)=(1+x^{-c})^{-k}$ and inserting it in Equation (4), the cdf of the GL-BIII distribution becomes

$$F(x) = \left\{1+\left[-\log\left(1-\left\{1-\left[1-\left(1+x^{-c}\right)^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}. \tag{19}$$

The pdf corresponding to Equation (19) is

$$\begin{aligned} f(x) &= \alpha\beta\,c\,k\,x^{-c-1}\left(1+x^{-c}\right)^{-k-1}\left[1-\left(1+x^{-c}\right)^{-k}\right]^{\alpha-1}\left\{1-\left[1-\left(1+x^{-c}\right)^{-k}\right]^{\alpha}\right\}^{\beta-1}\\ &\times\left[1-\left\{1-\left[1-\left(1+x^{-c}\right)^{-k}\right]^{\alpha}\right\}^{\beta}\right]^{-1}\left[-\log\left(1-\left\{1-\left[1-\left(1+x^{-c}\right)^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-2}\\ &\times\left\{1+\left[-\log\left(1-\left\{1-\left[1-\left(1+x^{-c}\right)^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-2}, \end{aligned} \tag{20}$$

where $\alpha>0$, $\beta>0$, $c>0$ and $k>0$ are shape parameters.Henceforth, X denotes a rv having density (20). The hrf of X can be expressed as

$$\begin{aligned} h(x) &= \alpha\beta\,c\,k\,x^{-c-1}\left(1+x^{-c}\right)^{-k-1}\left[1-\left(1+x^{-c}\right)^{-k}\right]^{\alpha-1}\left\{1-\left[1-\left(1+x^{-c}\right)^{-k}\right]^{\alpha}\right\}^{\beta-1}\\ &\times\left[1-\left\{1-\left[1-\left(1+x^{-c}\right)^{-k}\right]^{\alpha}\right\}^{\beta}\right]^{-1}\left[-\log\left(1-\left\{1-\left[1-\left(1+x^{-c}\right)^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-2}\\ &\times\left\{1+\left[-\log\left(1-\left\{1-\left[1-\left(1+x^{-c}\right)^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-2}\\ &\times\left[1-\left\{1+\left[-\log\left(1-\left\{1-\left[1-\left(1+x^{-c}\right)^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right]^{-1}. \end{aligned}$$

Figures 1 and 2 display some plots of the pdf and hrf of X for selected parameter values. Figure 1 reveals that the GL-BIII distribution is right-skewed, J, U and reversed-J shaped. Also, Figure 2 shows that the GL-BIII hrf can produce increasing, decreasing, bathtub, upside-down bathtub and increasing-decreasing-increasing shapes.

3.1 Expansion of the GL-BIII density

Recalling Equation (13), the cdf of the GL-BIII distribution follows as

$$F(x) = \frac{1}{v_0}\sum_{l=0}^{\infty} d_l\, G(x;c,(l+1)k). \tag{21}$$

By differentiating Equation (21), the GL-BIII density can be expressed as

$$f(x) = \frac{1}{v_0} \sum_{l=0}^{\infty} d_{l+1} \, g(x; c, (l+1)k), \tag{22}$$

where $g(x; c, (l+1)k)$ denotes the BIII density with shape parameters c and $(l+1)k$.Equation (22) reveals that the GL-BIII density is a linear combination of BIII densities, and then several of its mathematical properties can be obtained from those of the BIII distribution.

3.2 Properties of the GL-BIII distribution

Here, we present the ordinary and incomplete moments and mgf of X from Equation (22).

Proposition 3.1. *The nth ordinary moment of X can be expressed as*

$$\mu'_n \quad = \quad \mathbb{E}(X^n) = \frac{nk}{cv_0} \sum_{l=0}^{\infty} (l+1) \, d_{l+1} \, \frac{\Gamma\left(-\frac{n}{c}\right) \, \Gamma\left((l+1)k + \frac{n}{c}\right)}{\Gamma((l+1)k+1)}. \tag{23}$$

Proof. We can write from (22)

$$\mathbb{E}(X^n) = \frac{1}{v_0} \sum_{l=0}^{\infty} d_{l+1} \int_0^{\infty} x^n \, g(x; c, (l+1)k) \, dx = \frac{1}{v_0} \sum_{l=0}^{\infty} d_{l+1} \, \mathbb{E}\left(Y_{l+1}^n\right),$$

where Y_{l+1} ∼BIII$(c, (l+1)k)$. Based on the nth moment of the BIII model, the above result holds from

$$\mathbb{E}\left(Y_{l+1}^n\right) = \frac{n \, k \, (l+1) \, \Gamma\left(-\frac{n}{c}\right) \, \Gamma\left((l+1)k + \frac{n}{c}\right)}{c \, \Gamma((l+1)k+1)}.$$

□

Table 3 gives values for some basic quantities of the GL-BIII distribution: the first four ordinary moments, variance, standard deviation, coefficient of variation (CV) and the coefficients of skewness (CS) and kurtosis (CK). Table 4 reports well-known descriptive measures for this distribution based on quantiles. The skewness and kurtosis plots of X based on quantiles are displayed in Figure 3. These plots show that the parameters α and β play a significant role in modeling the skewness and kurtosis of X.

Proposition 3.2. *The nth incomplete moment of X is*

$$m_n(y) = \frac{\beta}{v_0} \sum_{l,j=0}^{\infty} \frac{(-1)^j \, d_{l+1}}{(n/c - j - 1)} \binom{-n/c - (l+1)k + 1}{j} G(y; c, (l+1)k)^{-n/c+j+1}. \tag{24}$$

Proof. We can write from (22)

$$m_n(y) = \int_0^y x^n \, f(x) dx = \frac{1}{v_0} \sum_{l=0}^{\infty} d_{l+1} \, m_{n,l}(y),$$

where $m_{n,l} = \int_0^y x^n \, g(x; c, (l+1)k) \, dx$. Using the qf of the BIII model and by changing $z = 1 - u^{1/(l+1)k}$, we can write

$$m_{n,l}(y) = \int_0^{G(y;c,(l+1)k)} \left[u^{-(l+1)k} - 1\right]^{-n/c} du = \beta \int_0^{G(y;c,(l+1)k)} z^{-n/c} \, (1-z)^{-n/c-(l+1)k+1} dz,$$

and then by expanding the binomial we arrive at (24). □

The Bonferroni and Lorenz curves of X (for a given probability π) are $B(\pi) = m_1(q)/(\pi \, mu'_1)$ and $L(\pi) = \pi \, B(\pi)$, respectively, where $q = Q(\pi)$ is the qf of X given by (27).

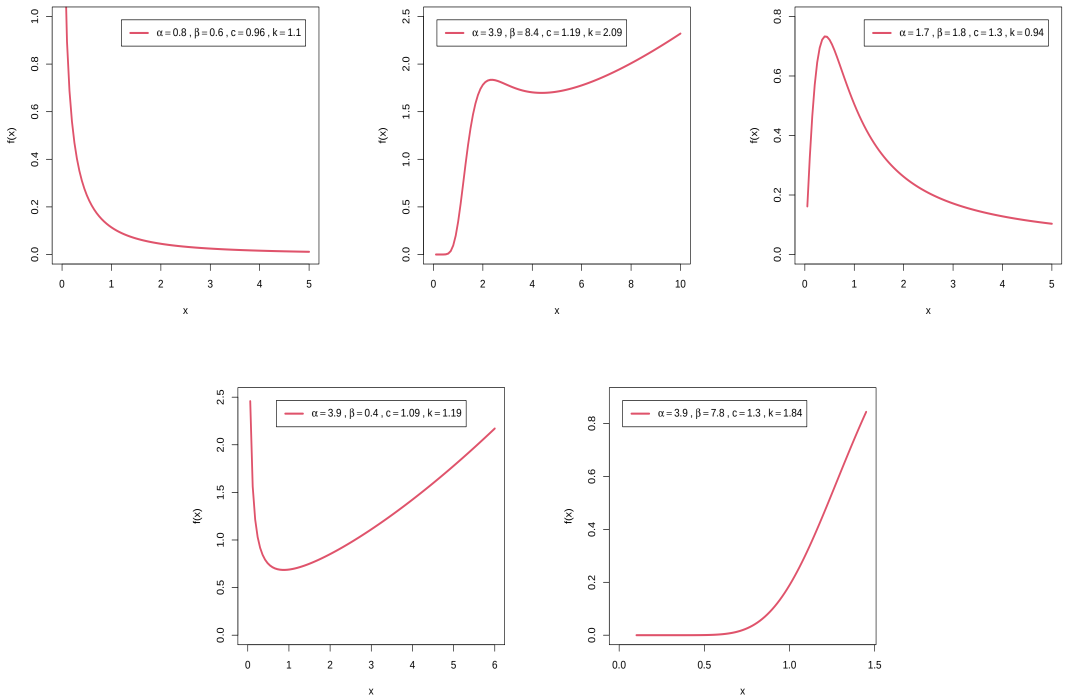


Figure 1: Plots of the GL-BIII density for some parameters.

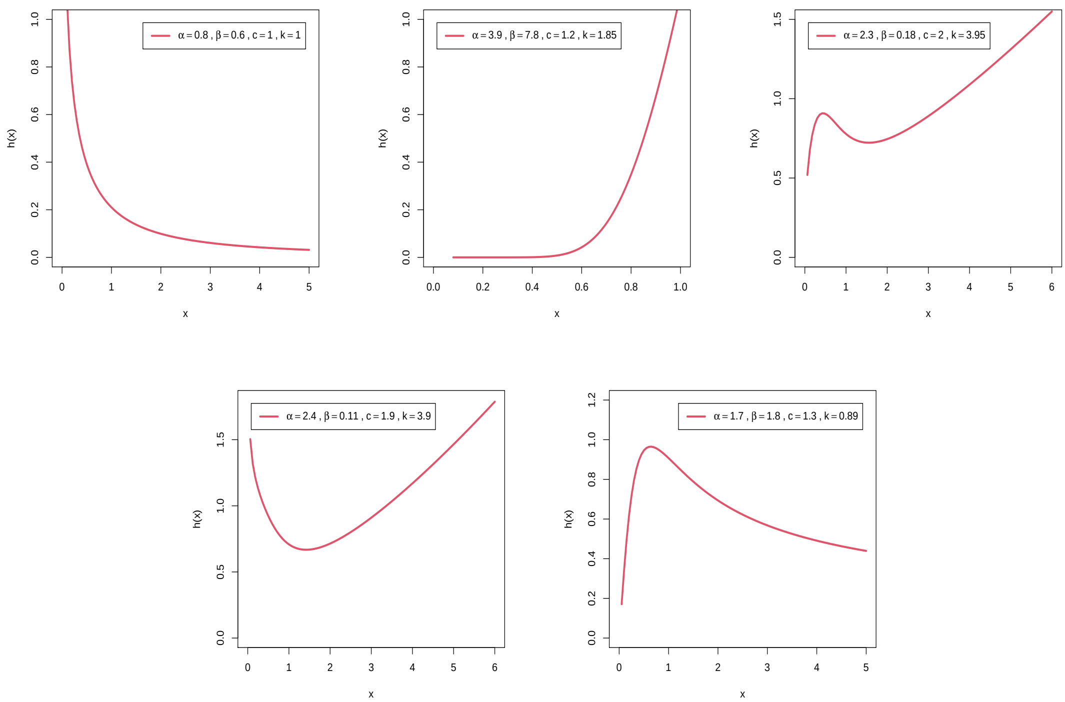


Figure 2: Plots of the GL-BIII hazard function for some parameters.

Table 3: Some quantities of the GL-BIII (α, β, c, k) distribution for selected parameters.

Parameters	$\mathbf{E}(X)$	$\mathbf{E}(X^2)$	$\mathbf{E}(X^3)$	$\mathbf{E}(X^4)$	$\mathbf{V}(X)$	$\sigma(X)$	CV	CS	CK
(1.5,0.1,1.5,2.5)	0.724	2.837	17.002	120.839	2.312	1.521	2.099	2.818	12.001
(1.5,0.2,1.5,2.5)	0.964	3.648	21.432	150.901	2.719	1.649	1.711	3.099	12.943
(1.5,0.5,1.5,2.5)	1.398	5.301	30.384	210.798	3.346	1.829	1.309	2.721	10.128
(1.5,1.9,1.5,2.5)	2.222	9.626	55.462	379.203	4.688	2.165	0.974	2.387	7.436
(2.5,0.3,1.5,2.5)	1.080	3.775	21.212	146.157	2.608	1.615	1.495	2.994	12.536
(2.5,2.3,1.5,2.5)	2.002	7.440	39.636	261.807	3.434	1.853	0.926	2.540	9.003

Table 4: Quantiles, Bowley skewness (B) and Moors kurtosis (M) of X for selected parameters.

Parameters	Q_1	Q_2	Q_3	B	M
(1.5,0.1,1.5,2.5)	0.027	0.273	1.888	0.735	7.314
(1.5,0.2,1.5,2.5)	0.162	0.618	2.808	0.655	7.005
(1.5,0.5,1.5,2.5)	0.563	1.294	4.524	0.631	7.011
(1.5,1.9,1.5,2.5)	1.658	2.982	8.854	0.632	7.133
(2.5,0.3,1.5,2.5)	0.280	0.736	2.255	0.539	3.735
(2.5,2.3,1.5,2.5)	1.259	2.002	4.359	0.521	3.844
(5.0,2.0,1.5,2.5)	0.824	1.313	2.639	0.461	2.847

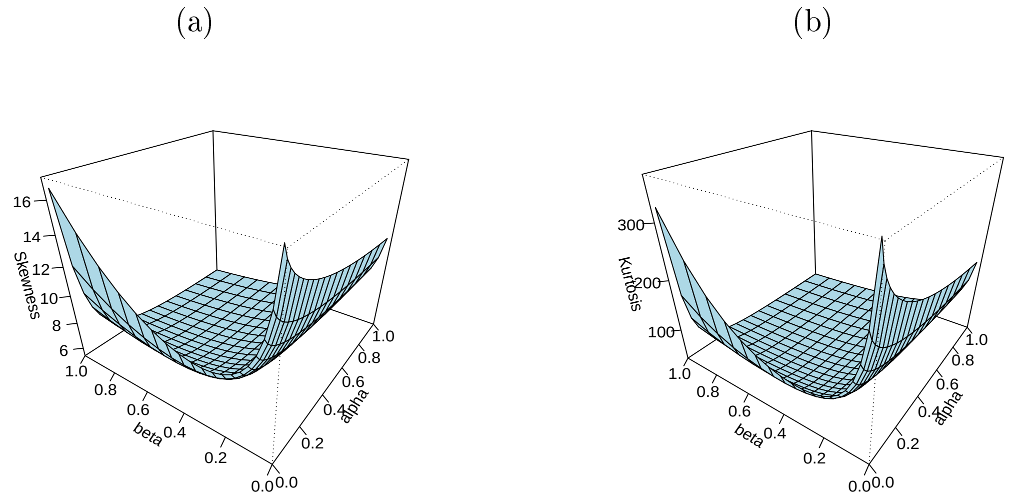


Figure 3: Plots of the skewness and kurtosis (based on quantiles) of X.

Proposition 3.3. *The mgf of X can be expressed as*

$$M_X(t) \quad = \quad \frac{c\,k}{v_0} \sum_{l=0}^{\infty} (l+1)\, d_{l+1} \sum_{p=0}^{\infty} \binom{-(l+1)k-1}{p} \Gamma[-(p+1)c]\,(-t)^{(p+1)c},$$

where $\Gamma(a) = \int_0^\infty t^{a-1}\,\mathrm{e}^{-t}dt$ is the gamma function.

Proof. It follows from (22)

$$M_X(t) = \mathbb{E}\left(\mathrm{e}^{\mathrm{tX}}\right) = \frac{1}{v_0}\sum_{l=0}^{\infty} d_{l+1}\int_0^{\infty} \mathrm{e}^{\mathrm{tx}}\,\mathrm{g}(\mathrm{x};\mathrm{c},(\mathrm{l}+1)\mathrm{k})\,\mathrm{dx}\,.$$
$$= \frac{c\,k}{v_0}\sum_{l=0}^{\infty}(l+1)\,d_{l+1}\int_0^{\infty} \mathrm{e}^{\mathrm{tx}}\mathrm{x}^{-\mathrm{c}-1}\,\left(1+\mathrm{x}^{-\mathrm{c}}\right)^{-(\mathrm{l}+1)\mathrm{k}-1}\mathrm{dx}. \tag{25}$$

Using the generalized binomial expansion,

$$\left(1+x^{-c}\right)^{-(l+1)k-1} = \sum_{p=0}^{\infty}\binom{-(l+1)k-1}{p}\,x^{-pc}. \tag{26}$$

Combining Equations (25) and (26), and solving the last integral, we obtain

$$M_X(t) = \frac{c\,k}{v_0}\sum_{l=0}^{\infty}(l+1)\,d_{l+1}\sum_{p=0}^{\infty}\binom{-(l+1)k-1}{p}\,\Gamma[-(p+1)\,c]\,(-t)^{(p+1)\,c}.$$

□

3.3 Quantile function and a simulation study

The qf of X is determined by inverting Equation (19)

$$Q(u) = \left[\left\{\left[1-\left\{1-\left[1-\mathrm{e}^{-\frac{\mathrm{u}}{1-\mathrm{u}}}\right]^{1/\beta}\right\}^{1/\alpha}\right]^{-1/k}\right\}-1\right]^{-1/c}. \tag{27}$$

We investigate the accuracy of the MLEs of the GL-BIII parameters using Monte Carlo simulations considering sample sizes $n = 100$, 200, 300 under three scenarios: I: $\alpha = 1.5$, $\beta = 0.8$, $c = 0.4$, and $k = 1.5$, II: $\alpha = 0.5$, $\beta = 1.8$, $c = 1.2$, and $k = 1.5$ and III: $\alpha = 2.5$, $\beta = 1.75$, $c = 0.8$, and $k = 0.75$. One thousand simulations are repeated for each sample size, and we calculate the average estimates (AEs), average biases and mean squared errors (MSEs) of the MLEs, namely

$$Bias(\hat{\theta}) = \sum_{i=1}^{N}\frac{\hat{\theta}_i}{N}-\theta \qquad \text{and} \qquad MSE(\hat{\theta}) = \sum_{i=1}^{N}\frac{(\hat{\theta}_i-\theta)^2}{N}.$$

The simulation results in Tables 5-7 clearly reveal that the biases and MSEs decrease when n increases. Then, the accuracy of the MLEs in estimating the parameters of the new distribution is evidently superior for large sample sizes as expected.

Table 5: Simulation results under scenario I.

	$n = 50$				$n = 100$			
	α	β	c	k	α	β	c	k
AE	1.522	0.755	0.419	1.616	1.605	0.862	0.379	1.771
Bias	0.522	-0.045	0.019	0.116	0.465	0.062	-0.019	0.171
MSE	0.172	0.050	0.012	0.060	0.102	0.097	0.006	0.009

	$n = 200$				$n = 500$			
	α	β	c	k	α	β	c	k
AE	1.512	0.823	0.384	1.584	1.503	0.794	0.408	1.498
Bias	0.112	0.023	-0.016	0.084	0.030	0.006	0.008	0.002
MSE	0.058	0.006	0.003	0.061	0.003	0.001	0.001	0.010

Table 6: Simulation results under scenario II.

	$n = 50$				$n = 100$			
	α	β	c	k	α	β	c	k
AE	0.997	2.135	0.974	2.038	0.746	1.941	0.976	1.871
Bias	0.957	0.335	-0.226	0.538	0.546	0.340	-0.224	0.371
MSE	2.878	0.346	0.197	0.871	0.763	0.347	0.347	0.398

	$n = 200$				$n = 500$			
	α	β	c	k	α	β	c	k
AE	0.672	1.887	1.008	1.693	0.535	1.823	1.181	1.538
Bias	0.472	0.287	-0.192	0.293	0.235	0.123	-0.069	0.138
MSE	0.703	0.246	0.123	0.274	0.329	0.080	0.048	0.107

Table 7: Simulation results under scenario III.

	$n = 50$				$n = 100$			
	α	β	c	k	α	β	c	k
AE	3.007	1.839	0.911	1.056	2.826	1.776	0.838	0.902
Bias	0.507	-0.411	0.111	0.306	0.326	-0.026	0.038	0.092
MSE	0.762	0.586	0.091	0.300	0.516	0.097	0.035	0.057

	$n = 200$				$n = 500$			
	α	β	c	k	α	β	c	k
AE	2.723	1.748	0.802	0.840	2.540	1.755	0.809	0.759
Bias	0.223	-0.107	0.002	0.090	0.040	-0.105	0.019	0.063
MSE	0.245	0.086	0.006	0.051	0.007	0.073	0.002	0.025

3.4 Estimation

Let $x_1, \cdots, x_n$ be n observations from the GL-BIII distribution (20) with parameter vector $\boldsymbol{\Theta} = (\alpha, \beta, c, k)^\top$. The log-likelihood $\ell = \ell(\boldsymbol{\Theta})$ for $\boldsymbol{\Theta}$ is

$$\begin{aligned}\ell &= n\log(\alpha\beta\, c\, k) + (\alpha-1)\sum_{i=1}^{n}\log\left(1-w^{1/\beta}\right)^{1/\alpha} + (\beta-1)\sum_{i=1}^{n}\log(w^{1/\beta}) - \sum_{i=1}^{n}\log(1-w) \\ &- 2\sum_{i=1}^{n}\log\left[-\log(1-w)\right] - 2\sum_{i=1}^{n}\log\left\{1+\left[-\log(1-w)\right]^{-1}\right\}. \qquad (28)\end{aligned}$$

The function ℓ can be easily maximized using the AdequacyModel package. The components of the score vector $U(\theta)$ are provided in Appendix A. The MLE $\widehat{\boldsymbol{\Theta}}$ of $\boldsymbol{\Theta}$ can also be obtained by solving the nonlinear equations $U_\alpha = 0$, $U_\beta = 0$, $U_c = 0$ and $U_k = 0$.

3.5 Analysis on flood and earthquake data sets: Univariate case

In this section, we compare the GL-BIII distribution with some well-known extended BIII distributions. We use two data sets representing different real life phenomenons to prove empirically that the new distribution provides better fits than its competitors. We compare it with the Beta-Dagum (BBIII) (Domma and Condino [34]), Kumaraswamy-Burr III (KwBIII) (Cordeiro *et al.* [35]), Gamma-Dagum (ZBGaBIII) (Oluyede *et al.* [36]), Marshall-Olkin extended Burr III (MOBIII) (Al-Saiari *et al.* [37]) and BIII distributions.We compute the log-likelihood function evaluated at the MLEs $(-\hat{\ell})$ using the method of a limited-memory quasi-Newton code for bound-constrained optimization (L-BFGS-B). For model comparison, a number of criterion go into assessing the scientific validity of the unknown model. The well-known goodness-of-fit (GoF) measures such as the Akaike information criterion (AIC), Corrected Akaike information criterion (CAIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), Anderson-Darling (A^*), Cramér–von Mises (W^*) and Kolmogrov-Smirnov (K-S) are adopted for model comparisons. Lower values of these statistics and higher p-values of the K-S statistic indicate better fits. The required computations are carried out using the `R` script `AdequacyModel` which is easily available from http://cran.r-project.org/web/packages/AdequacyModel/AdequacyModel.pdf.The first data set taken from [38] considers the maximum annual flood discharges (in units of 1,000 cubic feet per second) of the North Saskachevan River at Edmonton over a period of 48 years. The second data set represents the time intervals (in days) of successive earthquakes in Anatolya which were originally reported by [39]. The data set is taken from University of Bosphoros, Kandilli Observatory and Earthquake Research Institute-National Earthquake Monitoring Center (KOERI-NEMC, web address: http://www.koeri.boun.edu.tr).The numerical values of some statistics for the fitted models to these data sets are reported in Tables 8 and 9. Further, the MLEs and their standard errors (SEs in parentheses) for some fitted models are given in Tables 10 and 11. The figures in Tables 8 and 9

Table 8: The GoF statistics for some models fitted to data set 1.

Distribution	$-\hat{\ell}$	AIC	CAIC	BIC	HQIC	A^*	W^*	K-S	K-S p-value
GL-BIII	216.9624	441.9249	442.8551	449.4097	444.7534	0.2703	0.0358	0.0732	0.9590
BBIII	236.4482	450.8963	451.8265	456.3811	453.7248	0.5634	0.0833	0.0999	0.7247
KwBIII	231.5529	443.1059	443.6513	458.7195	455.2273	0.8001	0.1207	0.1121	0.5821
ZBGaBIII	221.3247	448.6495	449.1949	454.2631	450.7709	0.3599	0.0522	0.1258	0.4332
MOBIII	230.2612	466.5224	467.0679	472.1360	468.6438	0.4802	0.0709	0.2190	0.0202
BIII	233.5747	471.1493	471.4160	474.8917	472.5636	0.5527	0.3064	0.2157	0.0238

Table 9: The GoF statistics for some models fitted to data set 2.

Distribution	$-\hat{\ell}$	AIC	CAIC	BIC	HQIC	A^*	W^*	K-S	K-S p-value
GL-BIII	198.3868	404.7735	406.8788	409.4857	406.0237	0.3255	0.0375	0.0757	0.9654
BBIII	206.0734	438.1469	436.5537	444.5763	440.6756	0.4959	0.0869	0.1170	0.8574
KwBIII	200.6718	406.6119	408.8921	410.0072	410.8775	0.3992	0.0389	0.0783	0.9328
ZBGaBIII	260.2588	428.5775	428.9243	424.9464	421.0643	0.4174	0.1065	0.0837	0.7783
MOBIII	236.8419	429.6839	430.0906	436.1133	432.2126	0.3926	0.1135	0.1123	0.6866
BIII	268.5868	471.1737	465.3737	445.4549	442.8595	0.5307	0.1439	0.1553	0.6170

indicate that the GL-BIII model provides the best fit as compared to the other models. The plots in Figures 4-6 also support our claim.

4 BI-VARIATE EXTENSION OF GL-BIII DISTRIBUTION

Recently, there has been an increasing interest in developing bivariate models see the work in references [40]–[47]. Considering the two positive dependent random variables X, Y, the bivariate GL-BIII random vector (X, Y) with parameters $c, k, \alpha, \beta > 0$ and $-1 < \psi_1 + \psi_3 < 1, -1 < \psi_2 + \psi_3 < 1$, will be denoted

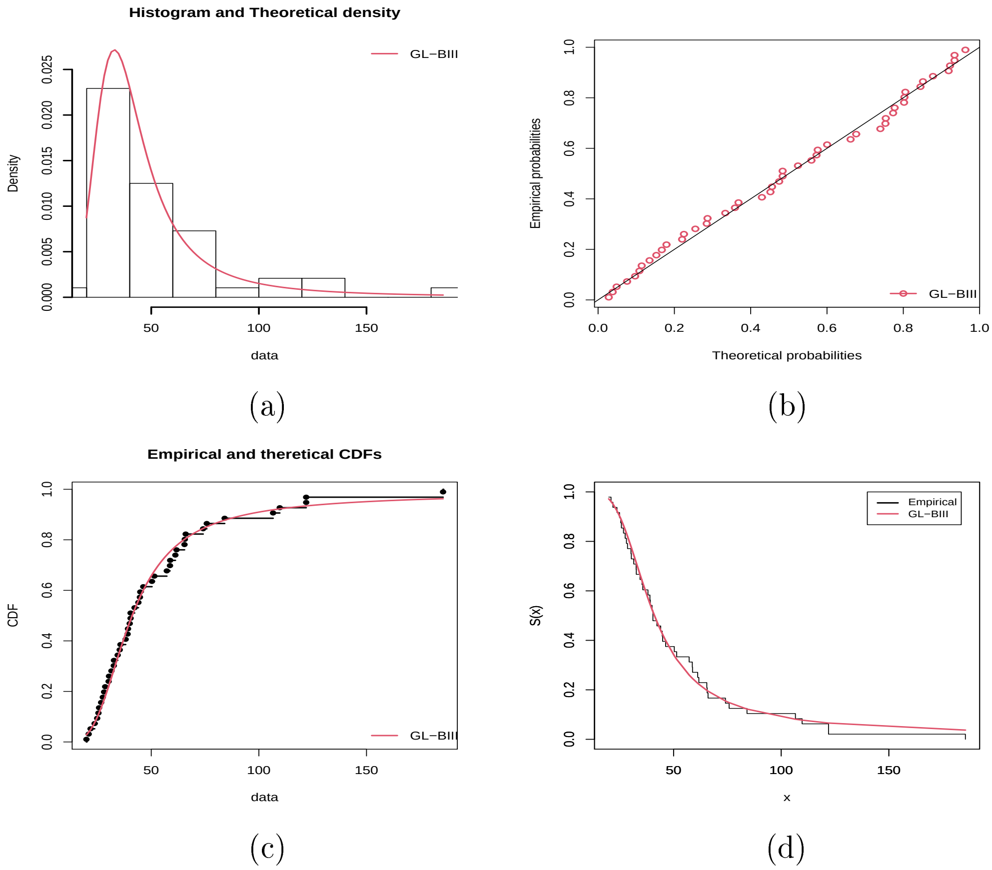


Figure 4: Estimated (a) density (b) PP-plot (c) cdf, and (d) sf plots for data set 1.

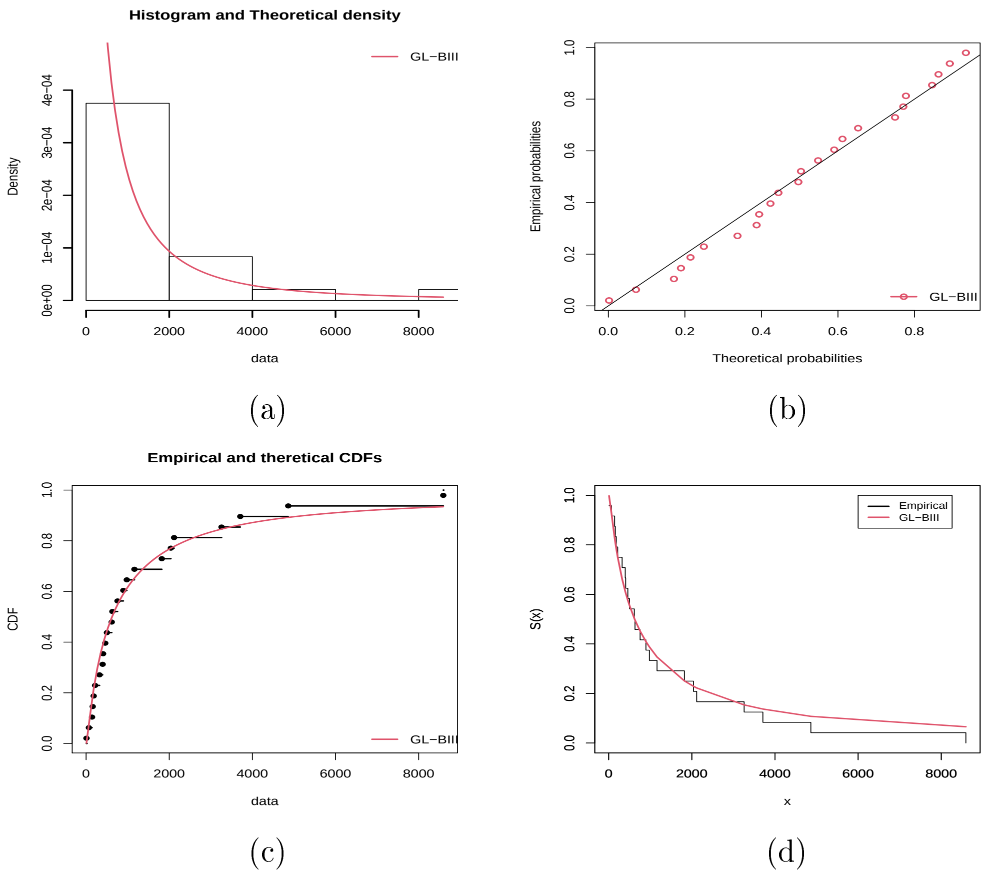
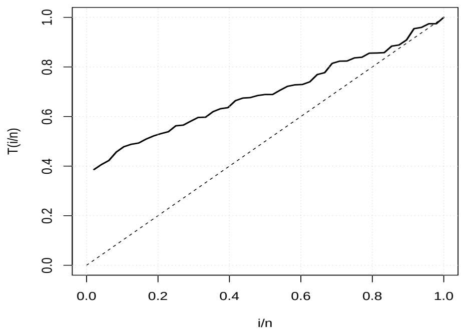
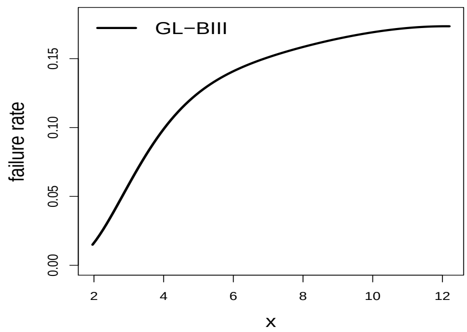
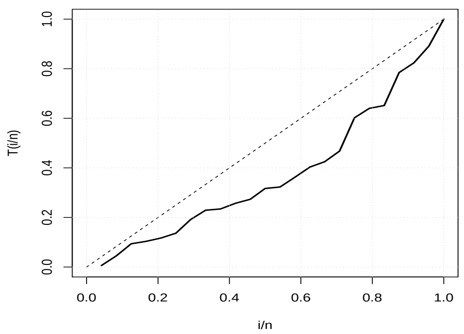


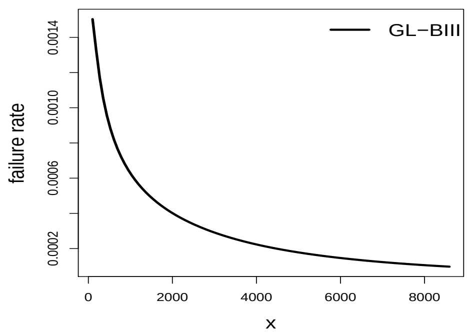
Figure 5: Estimated (a) density (b) PP-plot (c) cdf, and (d) sf plots for data set 2.





(a) (b)





(c) (d)

Figure 6: (a) TTT (b) Estimated hazard rate plots for data set 1 (c) TTT and (d) Estimated hazard rate plots for data set 2.

Table 10: Results from some fitted distributions to data set 1.

Distribution	α	β	c	k	λ	θ
GL-BIII	89.652	1.124	0.783	83.261	-	-
	(0.346)	(0.261)	(0.022)	(0.348)	-	-
BBIII	111.778	261.251	0.146	2.657	-	-
	(75.215)	(126.7369)	(0.032)	(1.131)	-	-
KwBIII	-	-	0.488	1.457	27.711	38.712
	-	-	(0.093)	(0.702)	(6.142)	(25.267)
ZBGaBIII	-	-	6.83	7.1940	-	24.240
	-	-	(0.007)	(0.095)	-	(0.704)
MOBIII	-	-	1.738	25.026	-	24.076
	-	-	(0.164)	(11.481)	-	(13.479)
BIII	-	-	1.135	52.038	-	-
	-	-	(0.086)	(14.676)	-	-

Table 11: Results from some fitted distributions to data set 2.

Distribution	α	β	c	k	λ	θ
GL-BIII	163.7293	0.2750	0.3724	76.5558	-	-
	(11.3982)	(0.0825)	(0.0509)	(0.1967)	-	-
BBIII	179.1883	1	3.035	0.692	-	-
	(125.353)	-	(6.682)	(0.314)	-	-
KwBIII	-	-	101.7721	19.2130	12.1118	1.6572
	-	-	(59.4673)	(7.293)	(4.1990)	(9.3824)
ZBGaBIII	-	-	22.089	12.891	-	21.169
	-	-	(0.003)	(0.113)	-	(0.573)
MOBIII	-	-	7.866	51.681	-	9.985
	-	-	(1.168)	(37.751)	-	(12.181)
BIII	-	-	5.378	22.046	-	-
	-	-	(0.514)	(11.697)	-	-

by $(X,Y) \sim$BIGL-BIII$(c,k,\alpha,\beta,\psi_1,\psi_2,\psi_3)$. The cdf $F(x,y)$, has the form

$$\begin{aligned} F(x,y) &= \left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1} \\ &\quad \times\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1} \\ &\quad \times\left(-\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right. \\ &\quad \times(\psi_1+\psi_3)+1+\psi_1+\psi_2+2\psi_3-(\psi_2+\psi_3) \\ &\quad \left.\times\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right). \end{aligned} \tag{29}$$

The pdf for the cdf 29 is

$$
\begin{aligned}
f(x,y) &= (\alpha\beta ck)^2(xy)^{-c-1}(1+x^{-c})(1+y^{-c})\left[1-(1+x^{-c})^{-k}\right]^{\alpha-1}\left[1-(1+y^{-c})^{-k}\right]^{\alpha-1} \\
&\times \left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1} \\
&\times \left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)^{-1}\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)^{-1} \\
&\times \left[-\log\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-2}\left[-\log\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-2} \\
&\times \left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-2} \\
&\times \left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-2} \\
&\times \left(-2(\psi_1+\psi_3)\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)+1+\psi_1+\psi_2 \\
&+2\psi_3-2(\psi_2+\psi_3)\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}. \qquad (30)
\end{aligned}
$$

The marginals for Eqs. 29 and 30 are, respectively, as

$$
\begin{aligned}
F_X(x) &= (1+\psi_1+\psi_3)\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1} \\
&\times -(\psi_1+\psi_3)\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-2}, \qquad (31)
\end{aligned}
$$

and

$$
\begin{aligned}
F_Y(y) &= (1+\psi_1+\psi_2)\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1} \\
&\times -(\psi_1+\psi_2)\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-2}, \qquad (32)
\end{aligned}
$$

with

$$
\begin{aligned}
f_X(x) &= \alpha\beta ckx^{-c-1}(1+x^{-c})\left[1-(1-x^{-c})^{-k}\right]^{\alpha-1}\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)^{-1} \\
&\times \left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}\left[-\log\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-2} \\
&\times \left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-2}(1+(\psi_1+\psi_3)) \\
&\times \left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
f_Y(y) &= \alpha\beta c k y^{-c-1}(1+y^{-c})\left[1-(1+y^{-c})^{-k}\right]^{\alpha-1}\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1} \\
&\times\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)^{-1}\left[-\log\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-2} \\
&\times\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-2} \\
&\times\left(1+(\psi_2+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)\right).
\end{aligned}
$$

4.1 Core properties

Here, we establish the core mathematical expressions for the bi-variate reliability function, hazard rate function, moments, incomplete moments and copula function of the BIGL - BIII distribution from Eq. (30) are as follows

Proposition 1. *Let $(X,Y)\sim BIGL-BIII(c,k,\alpha,\beta,\psi_1,\psi_2,\psi_3)$. Then the bivariate reliability function is given by*

$$
\begin{aligned}
R(x,y) &= 1-(1+\psi_1+\psi_3)\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1} \\
&\times-(\psi_1+\psi_3)\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-2} \\
&-(1+\psi_1+\psi_2)\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1} \\
&\times-(\psi_1+\psi_2)\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-2} \\
&+\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1} \\
&\times\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1} \\
&\times\left(-\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right) \\
&\times(\psi_1+\psi_3)+1+\psi_1+\psi_2+2\psi_3-(\psi_2+\psi_3) \\
&\times\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\Bigg).
\end{aligned}
$$

Proposition 2. *Let $(X,Y)\sim BIGL-BIII(c,k,\alpha,\beta,\psi_1,\psi_2,\psi_3)$. Then, the bivariate hazard rate func-*

tion, as in [48], is

$$
\begin{aligned}
h(x,y) &= \left[(\alpha\beta ck)^2(xy)^{-c-1}(1+x^{-c})(1+y^{-c})\left[1-(1+x^{-c})^{-k}\right]^{\alpha-1}\right]\left[1-(1+y^{-c})^{-k}\right]^{\alpha-1} \\
&\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1} \\
&\times\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)^{-1}\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)^{-1} \\
&\times\left[-\log\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-2}\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1} \\
&\times\left[-\log\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-2} \\
&\times\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-2} \\
&\times\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-2} \\
&\times\left(-2(\psi_1+\psi_3)\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right) \\
&+1+\psi_1+\psi_2+2\psi_3-2(\psi_2+\psi_3)\left[\left(\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)\right] \\
&\times\left[1-(1+\psi_1+\psi_3)\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right] \\
&\times-(\psi_1+\psi_3)\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-2} \\
&-(1+\psi_1+\psi_2)\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1} \\
&\times-(\psi_1+\psi_2)\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-2} \\
&+\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1} \\
&\times\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1} \\
&\times(\psi_1+\psi_3)\left(-\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right) \\
&+1+\psi_1+\psi_2+2\psi_3-(\psi_2+\psi_3) \\
&\times\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\Bigg)\Bigg]^{-1}.
\end{aligned}
$$

Proposition 3. *Let $(X,Y) \sim BIGL-BIII(c,k,\alpha,\beta,\psi_1,\psi_2,\psi_3)$. Then, the rth moment corresponding* *to density in (33) and (33), respectively, be as*

$$
\begin{aligned}
\mu_{X/Y}^{r}(y) &= E(X^r/Y=y)=\frac{rk}{cv_0}\sum_{l=0}^{\infty}(l+1)d_l\frac{\Gamma(-\frac{r}{c})\Gamma((l+1)k+\frac{r}{c})}{\Gamma((l+1)+1)} \\
&\times\left(1+\frac{\psi_1+\psi_3}{1+(\psi_2+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\{1-[1-(1+y^{-c})^{-k}]^{\alpha}\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)}\right) \\
&-\frac{2(\psi_1+\psi_3)}{v_0^2\left(1+(\psi_2+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\{1-[1-(1+y^{-c})^{-k}]^{\alpha}\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)\right)} \\
&\times\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}d_{l_1}d_{l_2}\delta_{l_1,l_2}(x),
\end{aligned}
$$

and

$$
\begin{aligned}
\mu_{Y/X}^{r}(x) &= E(Y^r/X=x)=\frac{rk}{cv_0}\sum_{l=0}^{\infty}(l+1)d_l\frac{\Gamma(-\frac{r}{c})\Gamma((l+1)k+\frac{r}{c})}{\Gamma((l+1)+1)} \\
&\times\left(1+\frac{\psi_2+\psi_3}{1+(\psi_1+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\{1-[1-(1+x^{-c})^{-k}]^{\alpha}\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)}\right) \\
&-\frac{2(\psi_2+\psi_3)}{v_0^2\left(1+(\psi_1+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\{1-[1-(1+x^{-c})^{-k}]^{\alpha}\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)\right)} \\
&\times\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}d_{l_1}d_{l_2}\delta_{l_1,l_2}(y),
\end{aligned}
$$

where

$$
\delta_{l_1,l_2}(\theta,\tau)=\int_0^{\infty}\theta^{\tau}G(\theta;c,(l_1+1)k)g(\theta;c,(l_2+1)d\theta \tag{33}
$$

and $G(.), g(.)$ are given by 21 and 22, respectively.

Proposition 4. *Let $(X,Y)\sim BIGL-BIII(c,k,\alpha,\beta,\psi_1,\psi_2,\psi_3)$. Then the incomplete moments corresponding to BIGL-BIII model are*

$$
\begin{aligned}
\mu_{r,s} &= E(X^rY^s)=(1+\psi_1+\psi_2+2\psi_3)\frac{rsk^2}{c^2v_0^2}] \\
&\times\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}(l_1+1)(l_2+1)d_{l_1}d_{l_2}\frac{\Gamma(-\frac{r}{c})\Gamma((l_1+1)k+\frac{r}{c})}{\Gamma((l_1+1)+1)}\frac{\Gamma(-\frac{s}{c})\Gamma((l_2+1)k+\frac{s}{c})}{\Gamma((l_2+1)+1)} \\
&-2\left((\psi_1+\psi_3)\sum_{l_2=0}^{\infty}(l_2+1)d_{l_2}\delta_{l_1,l_2}(x,r)\frac{\Gamma(-\frac{s}{c})\Gamma((l_2+1)k+\frac{s}{c})}{\Gamma((l_2+1)+1)}\right) \\
&+\left((\psi_2+\psi_3)\sum_{l_1=0}^{\infty}(l_1+1)d_{l_1}\delta_{l_1,l_2}(y,s)\frac{\Gamma(-\frac{r}{c})\Gamma((l_1+1)k+\frac{r}{c})}{\Gamma((l_1+1)+1)}\right),
\end{aligned}
$$

where $\delta_{l_1,l_2}(\theta,\tau)$ is given by 33.

Proposition 5. *Let $(X,Y)\sim BIGL-BIII(c,k,\alpha,\beta,\psi_1,\psi_2,\psi_3)$. Then its copula function due to [49],* *is given by*

$$\begin{aligned} c(u,v) &= \left[1+(\psi_1+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)\right] \\ &+ \left[(\psi_2+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)\right] \\ &\times \left[1+(\psi_1+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+x^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)\right]^{-1} \\ &\times \left[1+(\psi_2+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1+y^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)\right]^{-1}. \end{aligned}$$

4.2 Related estimation results

The maximum log-likelihood function $L(.)$ for a random sample from $(X,Y) \sim$ BIGL-BIII$(c,k,\alpha,\beta,\psi_1,\psi_2,\psi_3)$ is

$$\begin{aligned} & 2n\ln(\alpha\beta ck) - (c+1)\sum_{i=1}^{n}\{\ln(x_i)+\ln(y_i)\} + \sum_{i=1}^{n}\{\ln(1-x_i^{-c})+\ln(1-y_i^{-c})\} \\ & +(\alpha-1)\sum_{i=1}^{n}\{\ln[1-(1-x_i^{-c})^{-k}]+\ln[1-(1-y_i^{-c})^{-k}]\} \\ & +(\beta-1)\sum_{i=1}^{n}\left\{\ln\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}+\ln\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}\right\} \\ & -\sum_{i=1}^{n}\left\{\ln\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)+\ln\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right\} \\ & -2\sum_{i=1}^{n}\left\{\ln\left[-\log\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]+\ln\left[-\log\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]\right\} \\ & +\sum_{i=1}^{n}\left\{\ln\left[1+(\psi_1+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)\right]\right\} \\ & +\left\{\left[(\psi_2+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)\right]\right\}. \end{aligned} \tag{34}$$

The maximum likelihood estimators of parameters can be obtained by numerically solving the system of non- linear equations, presented in Appendix B.

4.3 Numerical illustrations of BIGL-BIII

Here we investigate three different examples of the $BIGL-BIII(c,k,\alpha,\beta,\psi_1,\psi_2,\psi_3)$, to show the flexibility of the new model and its applicability to different types of bivariate data including ill-conditioned data.

4.3.1 Example 1

Consider that $(X,Y) \sim$ BIGL-BIII$(1.9, 2.2, .4, 1.1, -0.7, -0.5, 0.2)$ parameter combination which are totally random, then the characteristics of its distribution can be observed in Fig. 7. We note that the model can be applicable to data with unimodel and high left skewness, see Fig 7(a). The cdf (Fig. 7(b)) approaches one for high value of x and y that means that it is suitable to model big data with no need to transform. The bivariate hazard function is given in Fig. 7(c). The independence structure is clear in Figs. 7(d) & 7(e). The high skewness of the marginals can easily be observed in Figs. 7(f) & 7(h). The availability of the model to fit large numbered data is clear in Figs. 7(g) and 7(i). This example shows that, the model can fit data with high values and high right skewness.

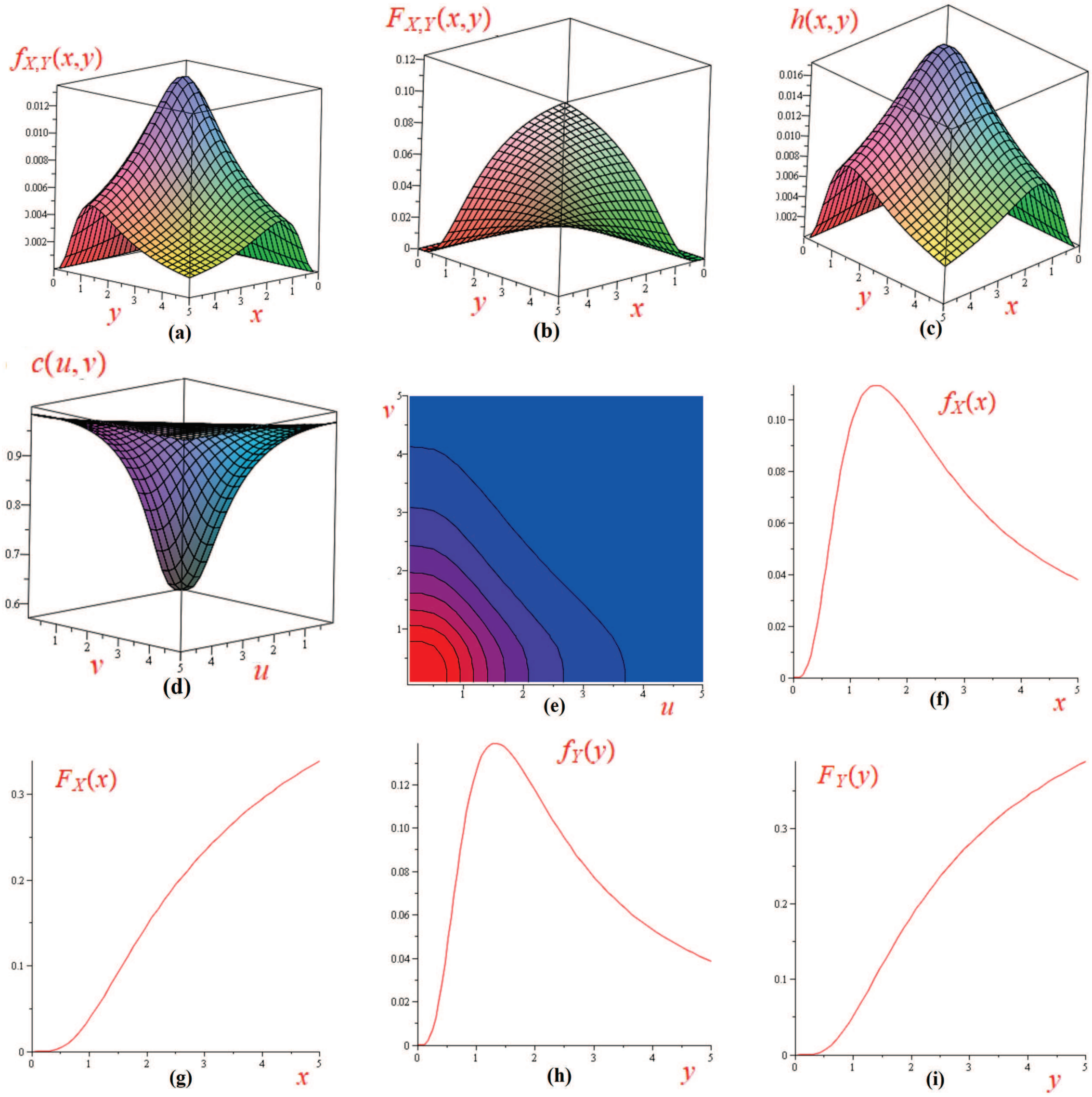


Figure 7: Surface plots of the characteristics of BIGL-BIII model at $(1.9, 2.2, .4, 1.1, -0.7, -0.5, 0.2)$ parameter values.

4.3.2 Example 2

Let us here consider that $(X, Y) \sim$ BIGL-BIII$(1.3, 0.9, 2.5, 0.9, 0.2, -0.9, 0.6)$. In contrast to example 1, this part illustrates that the model is suitable to fit the rational data, in $(0, 1)$ Figs 8(a) and 8(b) with completely different properties of hazard function Fig. 8(c) and dependence structures, Figs. 8(d) and 8(e). The unimodality of marginals appears near zero, Figs. 8(f) and 8(h), that suggests the applicability of the model to ill-conditioned data. When $(x, y) \to (1, 1)$ the cdfs approach 1, Figs 8(g) and 8(i), that ensure the fully description of rational data.

4.3.3 Example 3

Let $(X, Y) \sim$ BIGL-BIII$(2.6, 1.8, 3.4, 1.7, 0.6, -0.5, -0.3)$. This example presents a distribution with different set of properties in comparison with example one and two, Fig. 7, Fig. 8 and Fig. 9. The unimodality can be observed in Fig. 9(a) with moderate skewness. The model fits data with moderate

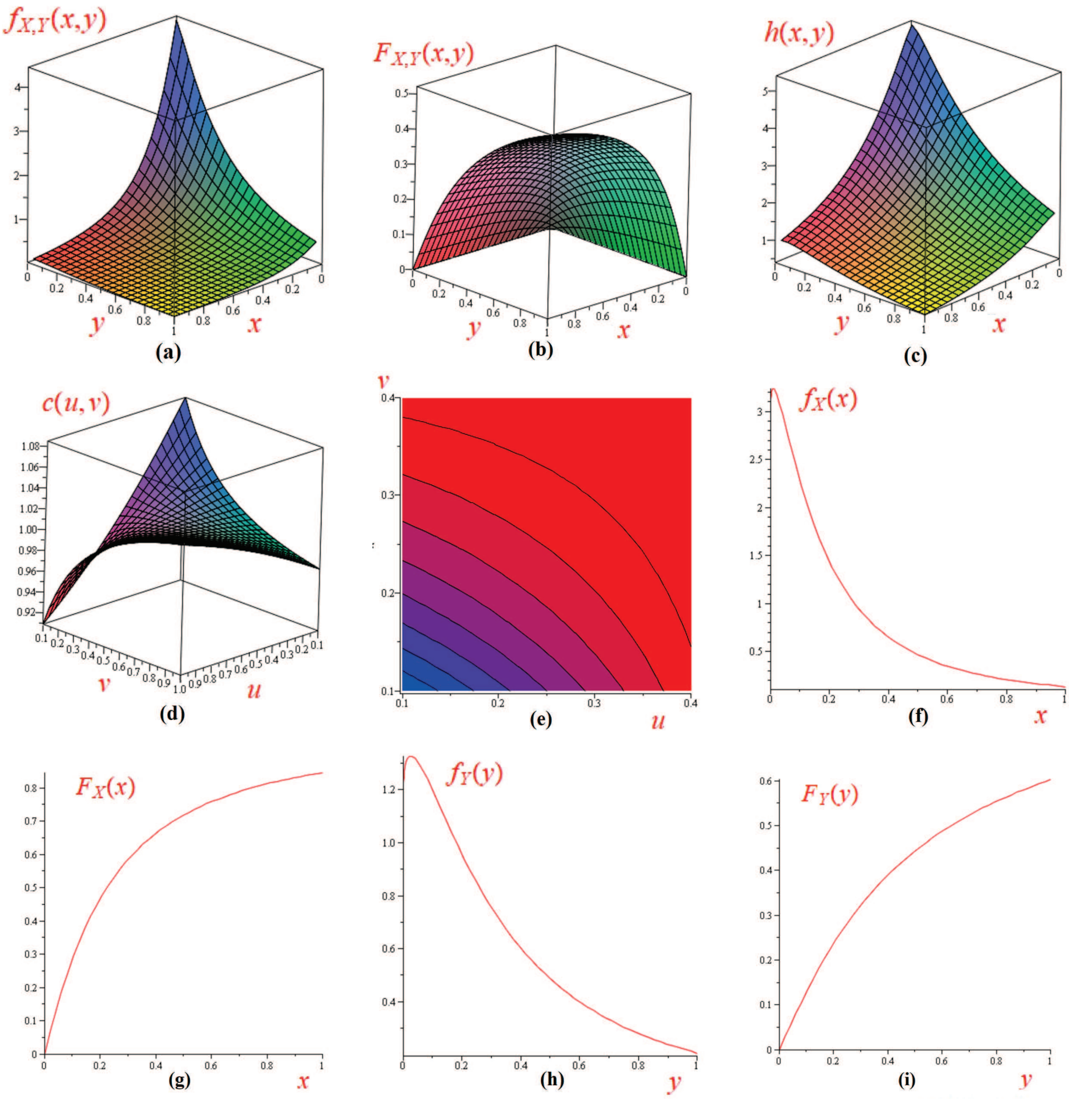


Figure 8: Plots of the characteristics of BIGL-BIII model at $(1.3, 0.9, 2.5, 0.9, 0.2, -0.9, 0.6)$ parameter values.

values $1 < x, y < 10$, Fig. 9(b) with different hazard function and independence structure Figs. 9(d) and 9(e). The moderate skewness of marginals can be observed in Figs. 9(f) and 9(h). The model also fits well-conditioned data, Figs 9(g) and 9(i).

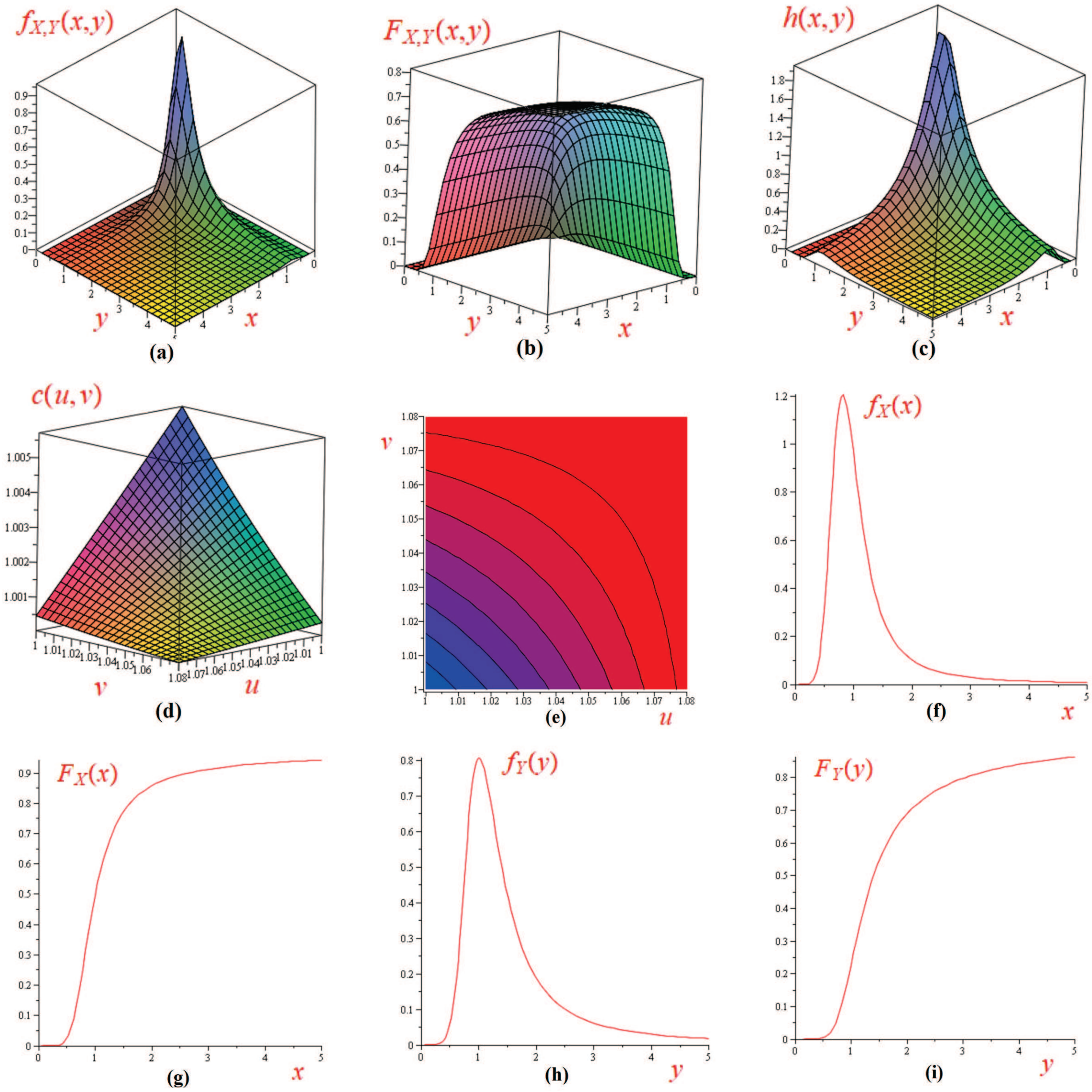


Figure 9: Plots of the characteristics of BIGL-BIII model at $(2.6, 1.8, 3.4, 1.7, 0.6, -0.5, -0.3)$ parameter values.

4.4 Real life application: Bivariate case

In this section, this model is applied to life time data simulated by [50] and investigated by [51]. The data represents the operational lifespan of the components, *viz. a viz. the processor lifetime and the memory lifetime* of central processing unit (CPU) of a computer. The authors in [50] studied this data in the context of two-component system reliability configuration. They simulated the data in terms of latent variable using Marshall-Olkin bivariate Exponential distribution. Let X and Y represent the processor lifetime and the memory lifetime, respectively. The parameters of the model $(X, Y) \sim$ BIGL-BIII$(c, k, \alpha, \beta, \psi_1, \psi_2, \psi_3)$ are estimated in Table. 12 and its distribution functions are given in Fig. 10.

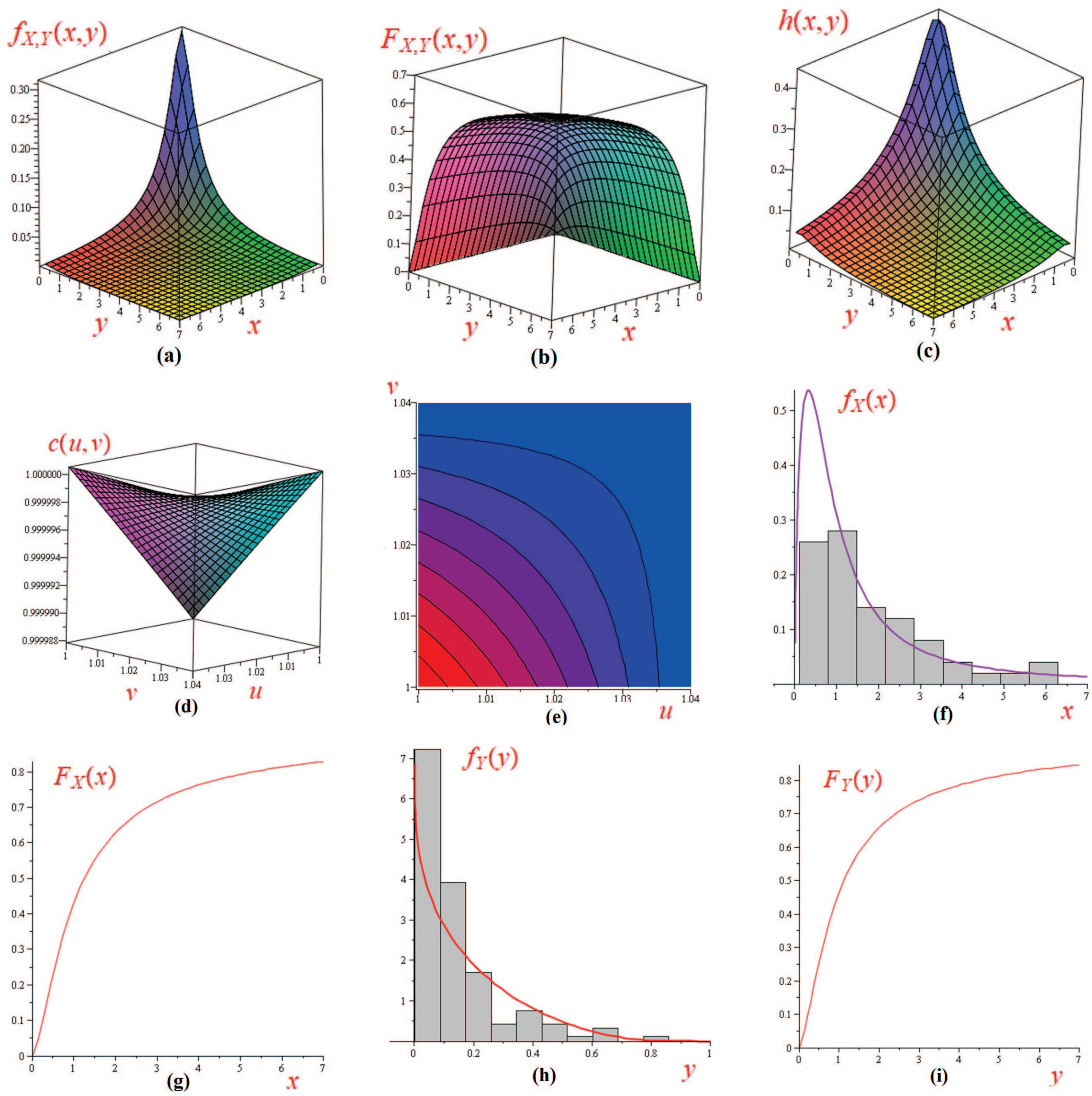


Figure 10: Plots of the fitted characteristics of BIGL-BIII model on lifetime data.

Table 12: The estimates of the unknown parameters generated from the BIGL-BIII class.

Estimates (c,k,α,β,ψ_1,ψ_2,ψ_3)	AIC	BIC
1.416, 4.310, 3.203, 0.259, -0.927, -0.788, 0.693	376.710	390.094

5 Concluding remarks

A new family of distributions so called *generalized logistic-G* (GL-G) has been established using a distinct generator function, having two additional parameters. The theoretical foundation of GL-G has been established using structural properties. The inference related to model parameters has been achieved adopting the framework of maximum likelihood estimation. For bounded parameter interval, inverse-Burr (BIII) distribution has been chosen to study one of the special model arising due to GL-G class called the *generalized logistic-Burr-III* (GL-BIII) distribution. The convergence of parameter estimates has been checked using Monte-carlo simulation technique. For univariate case, two real life data sets related to extreme event occurrences has been used to illustrate the potentiality of this special model in comparison to five well-established families. Further, a bi-variate extension of the GLBIII termed as *Bivariate generalized logistic-Burr-III* (BIGL-BIII) is established with a theoretical and applied perspective. Given the flexibility of the proposed class of distribution, an extended regression model based on the logarithm of the random variable can be establish in future. The inference related to parameter estimation of the GL-BIII model parameter can be carried out based on different method of estimations. We hope that the proposed model can attract the attention of applied practitioners from various fields.

AUTHOR INFORMATION

Author Contributions

SADAF KHAN: Conceptualization, Methodology, Writing - Original Draft. **FARRUKH JAMAL**: Resources, Supervision, Data Acquisition **MUHAMMAD HUSSAIN TAHIR**: Writing - Review & Editing . **AHMED ELHASSANEIN**: Data Analysis, Visualization, Writing - Review & Editing .

Funding

This research received no specific grant from any funding agency.

ACKNOWLEDGMENTS

We thank colleagues for their valuable feedback and anonymous reviewers for their constructive comments.

ABBREVIATIONS

Transformed-Transformer	T-X
Probability density function	pdf
Cumulative distribution function	cdf
Random variable	rv
Survival function	sf
Generalized logistic	GL
Burr III	BIII
Generalized logistic-Burr-III	GLBIII
Hazard rate function	hrf
Quantile function	qf
Exponentiated-G	exp-G
Moment generating function	mgf
Average estimates	AEs
Average biases	Bias
Mean squared errors	MSEs
Beta-Dagum	BBIII
Kumaraswamy-Burr III	KwBIII
Gamma-Dagum	ZBGaBIII
Marshall-Olkin extended Burr III	MOBIII
Bound-constrained optimization	L-BFGS-B
Goodness-of-fit	GoF
Akaike information criterion	AIC
Corrected Akaike information criterion	CAIC
Bayesian Information Criterion	BIC
Hannan-Quinn Information Criterion	HQIC
Anderson-Darling	A^*
Cramér–von Mises	W^*
Kolmogrov-Smirnov	K-S
Standard errors	SEs
Inverse-Burr	BIII
Bivariate generalized logistic-Burr-III	BIGL-BIII

Appendix A

$$
\begin{aligned}
U_\alpha \;=\;& \frac{n}{\alpha} + \alpha\sum_{i=1}^{n}\log(1-w^{1/\beta})\,w_{i\alpha} - \sum_{i=1}^{n}\left(1-w^{1/\beta}\right) + (\beta-1)\sum_{i=1}^{n}\frac{w_{i\alpha}}{w^{1/\beta}} + \sum_{i=1}^{n}\frac{w_{i\alpha}}{1-w} \\
& -2\sum_{i=1}^{n}\frac{w_{i\alpha}}{(1-w)[-\log(1-w)]} + 2\sum_{i=1}^{n}\frac{(1-w)w_{i\alpha}}{[-\log(1-w)]^2\,[1+\{-\log(1-w)\}^{-1}]}\,,
\end{aligned}
$$

$$
\begin{aligned}
U_\beta \;=\;& \frac{n}{\beta} + \left(\frac{\alpha-1}{\alpha}\right)\sum_{i=1}^{n}\frac{w^{1/\beta}\log(w)\,w_{i\beta}}{\left(1-w^{1/\beta}\right)^{1/\alpha}} + \beta\sum_{i=1}^{n}[\log(w)\,w_{i\beta}] - \sum_{i=1}^{n}\log(w^{1/\beta}) \\
& + \sum_{i=1}^{n}\frac{w_{i\beta}}{(1-w)} - 2\sum_{i=1}^{n}\frac{w_{i\beta}}{(1-w)[-\log(1-w)]}\,,
\end{aligned}
$$

$$
\begin{aligned}
U_c \;=\;& \left(\frac{1-\alpha}{\alpha\,\beta}\right)\sum_{i=1}^{n}\left[\frac{w^{(1/\beta)-1}\,w_{i\,c}}{(1-w^{1/\beta})}\right] + \left(\frac{1-\beta}{\beta}\right)\sum_{i=1}^{n}\frac{w_{i\,c}}{w} - 2\sum_{i=1}^{n}\frac{w_{i\,c}}{[-\log(1-w)](1-w)} \\
& + \sum_{i=1}^{n}\frac{w_{i\,c}}{(1-w)} + 2\sum_{i=1}^{n}\frac{(1-w)^{-1}w_{i\,c}}{[1+\{-\log(1-w)\}^{-1}]\,[-\log(1-w)]^2}\,,
\end{aligned}
$$

$$
\begin{aligned}
U_k \;=\;& \left(\frac{1-\alpha}{\alpha\,\beta}\right)\sum_{i=1}^{n}\left[\frac{w^{(1/\beta)-1}\,w_{i\,k}}{(1-w^{1/\beta})}\right] + \left(\frac{\beta-1}{\beta}\right)\sum_{i=1}^{n}\frac{w_{i\,k}}{w} - 2\sum_{i=1}^{n}\frac{w_{i\,k}}{[-\log(1-w)](1-w)} \\
& + \sum_{i=1}^{n}\frac{w_{i\,k}}{(1-w)} + 2\sum_{i=1}^{n}\frac{(1-w)^{-1}w_{i\,k}}{[1+\{-\log(1-w)\}^{-1}]\,[-\log(1-w)]^2}\,,
\end{aligned}
$$

$$
\begin{gathered}
\text{where } w_i = \left\{1-\left[1-(1+x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}, \\
w_{i\alpha} = \beta\left\{1-\left[1-(1+x_i^{-c})^{-k}\right]^{\alpha}\right\}\log\left[1-\left(1+x_i^{-c}\right)^{-k}\right]\left\{1-\left[1-(1+x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}, \\
w_{i\beta} = \left\{1-\left[1-(1+x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\log\left\{1-\left[1-(1+x_i^{-c})^{-k}\right]^{\alpha}\right\}, \\
w_{i\,c} = \alpha\beta\,kx_i^{-c}\log(x_i)\left(x_i^{-c}+1\right)^{-k-1}\left[1-(1+x_i^{-c})^{-k}\right]^{\alpha-1}\left\{1-\left[1-(1+x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}, \\
w_{i\,k} = \alpha\beta\left\{-\left(1+x_i^{-c}\right)^{-k}\right\}\log\left(1+x_i^{-c}\right)\left[1-(1+x_i^{-c})^{-k}\right]^{\alpha-1}\left\{1-\left[1-(1+x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}.
\end{gathered}
$$

Appendix B

$$
\begin{aligned}
L_c \;=\;& \frac{2n}{c} - \sum_{i=1}^{n} \{\ln(x_i) + \ln(y_i)\} - \sum_{i=1}^{n} \left\{ \frac{x_i^{-c}\ln(x_i)}{1-x_i^{-c}} + \frac{y_i^{-c}\ln(y_i)}{1-y_i^{-c}} \right\} \\
& + k(\alpha-1)\sum_{i=1}^{n} \left\{ \frac{x_i^{-c}(1-x_i^{-c})^{-k-1}\ln(x_i)}{1-(1-x_i^{-c})^{-k}} + \frac{y_i^{-c}(1-y_i^{-c})^{-k-1}\ln(y_i)}{1-(1-y_i^{-c})^{-k}} \right\} + \alpha k(\beta-1) \\
& \times \sum_{i=1}^{n} \left\{ \frac{x_i^{-c}(1-x_i^{-c})^{-k-1}[1-(1-x_i^{-c})^{-k}]^{\alpha-1}\ln(x_i)}{1-[1-(1-x_i^{-c})^{-k}]^{\alpha}} + \frac{y_i^{-c}(1-y_i^{-c})^{-k-1}[1-(1-y_i^{-c})^{-k}]^{\alpha-1}\ln(y_i)}{1-[1-(1-y_i^{-c})^{-k}]^{\alpha}} \right\} \\
& - \alpha k\beta \sum_{i=1}^{n} \left\{ \frac{x_i^{-c}(1-x_i^{-c})^{-k-1}[1-(1-x_i^{-c})^{-k}]^{\alpha-1}\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}\ln(x_i)}{\left\{1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right\}} \right\} \\
& \left\{ + \frac{y_i^{-c}(1-y_i^{-c})^{-k-1}[1-(1-y_i^{-c})^{-k}]^{\alpha-1}\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}\ln(y_i)}{\left\{1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right\}} \right\} \\
& + \alpha k\beta \sum_{i=1}^{n} \left\{ \frac{x_i^{-c}(1-x_i^{-c})^{-k-1}[1-(1-x_i^{-c})^{-k}]^{\alpha-1}\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}\ln(x_i)}{\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\log\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)} \right\} \\
& + \left\{ + \frac{y_i^{-c}(1-y_i^{-c})^{-k-1}[1-(1-y_i^{-c})^{-k}]^{\alpha-1}\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}\ln(y_i)}{\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\log\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)} \right\} \\
& + \alpha k\beta \sum_{i=1}^{n} \left\{ \left[1 + (\psi_1+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right) \right] \right\} \\
& \left[(\psi_2+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right) \right]^{-1} \\
& \times \left[4(\psi_1+\psi_3)\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-2} \right] \\
& \times \left[-\log\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-2} \\
& \times \frac{x_i^{-c}(1-x_i^{-c})^{-k-1}[1-(1-x_i^{-c})^{-k}]^{\alpha-1}\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}\ln(x_i)}{\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)} \\
& + 4(\psi_2+\psi_3)\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-2} \\
& \times \left[-\log\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-2} \\
& \times \left\{ \left[\frac{y_i^{-c}(1-y_i^{-c})^{-k-1}[1-(1-y_i^{-c})^{-k}]^{\alpha-1}\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}\ln(y_i)}{\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)} \right] \right\},
\end{aligned}
$$

$$
\begin{aligned}
L_\alpha \;=\; & \frac{2n}{\alpha} + \sum_{i=1}^{n} \left\{ \ln[1-(1-x_i^{-c})^{-k}] + \ln[1-(1-y_i^{-c})^{-k}] \right\} \\
& +(\beta-1)\sum_{i=1}^{n}\left\{ \frac{\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\ln\left[1-(1-x_i^{-c})^{-k}\right]}{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}} + \frac{\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\ln\left[1-(1-y_i^{-c})^{-k}\right]}{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}} \right\} \\
& +\beta\sum_{i=1}^{n}\left\{ \frac{\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\ln\left[1-(1-x_i^{-c})^{-k}\right]}{1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}} \right\} \\
& +\left\{ \frac{\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\ln\left[1-(1-y_i^{-c})^{-k}\right]}{1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}} \right\} \\
& +\beta\sum_{i=1}^{n}\left\{ \frac{[1-(1-x_i^{-c})^{-k}]^{\alpha}\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}\ln[1-(1-x_i^{-c})^{-k}]}{\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\log\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)} \right\} \\
& +\left\{ \frac{[1-(1-y_i^{-c})^{-k}]^{\alpha}\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}\ln[1-(1-y_i^{-c})^{-k}]}{\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\log\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)} \right\} \\
& +\beta\sum_{i=1}^{n}\left\{\left[1+(\psi_1+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)\right]\right\} \\
& \left[(\psi_2+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)\right]^{-1} \\
& \times\left[4(\psi_1+\psi_3)\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-2}\right] \\
& \times\left[-\log\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-2} \\
& \times\frac{[1-(1-x_i^{-c})^{-k}]^{\alpha}\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}\ln[1-(1-x_i^{-c})^{-k}]}{\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)} \\
& +4(\psi_2+\psi_3)\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-2} \\
& \times\left[-\log\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-2} \\
& \times\left\{\left[\frac{[[1-(1-y_i^{-c})^{-k}]^{\alpha}\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}\ln[1-(1-y_i^{-c})^{-k}]}{\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)}\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
L_k &= \frac{2n}{k} + (\alpha-1)\sum_{i=1}^{n}\left\{\frac{(1-x_i^{-c})^{-k}\ln(1-x_i^{-c})}{1-(1-x_i^{-c})^{-k}} + \frac{(1-y_i^{-c})^{-k}\ln(1-y_i^{-c})}{1-(1-y_i^{-c})^{-k}}\right\} + \alpha(\beta-1) \\
&\times \sum_{i=1}^{n}\left\{\frac{(1-x_i^{-c})^{-k}[1-(1-x_i^{-c})^{-k}]^{\alpha-1}\ln(1-x_i^{-c})}{1-[1-(1-x_i^{-c})^{-k}]^{\alpha}} + \frac{(1-y_i^{-c})^{-k}[1-(1-y_i^{-c})^{-k}]^{\alpha-1}\ln(1-y_i^{-c})}{1-[1-(1-y_i^{-c})^{-k}]^{\alpha}}\right\} \\
&-\alpha\beta\sum_{i=1}^{n}\left\{\frac{(1-x_i^{-c})^{-k}[1-(1-x_i^{-c})^{-k}]^{\alpha-1}\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}\ln(1-x_i^{-c})}{\{1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\}}\right\} \\
&\left\{+\frac{(1-y_i^{-c})^{-k}[1-(1-y_i^{-c})^{-k}]^{\alpha-1}\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}\ln(1-y_i^{-c})}{\{1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\}}\right\} \\
&+\alpha\beta\sum_{i=1}^{n}\left\{\frac{(1-x_i^{-c})^{-k}[1-(1-x_i^{-c})^{-k}]^{\alpha-1}\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}\ln(1-x_i^{-c})}{\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\log\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)}\right\} \\
&+\left\{+\frac{(1-y_i^{-c})^{-k}[1-(1-y_i^{-c})^{-k}]^{\alpha-1}\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}\ln(1-y_i^{-c})}{\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\log\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)}\right\} \\
&+\alpha\beta\sum_{i=1}^{n}\left\{\left[1+(\psi_1+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)\right]\right\} \\
&\left[(\psi_2+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)\right]^{-1} \\
&\times\left[4(\psi_1+\psi_3)\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-2}\right] \\
&\times\left[-\log\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-2} \\
&\times\frac{(1-x_i^{-c})^{-k}[1-(1-x_i^{-c})^{-k}]^{\alpha-1}\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}\ln(1-x_i^{-c})}{\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)} \\
&+4(\psi_2+\psi_3)\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-2} \\
&\times\left[-\log\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-2} \\
&\times\left\{\left[\frac{(1-y_i^{-c})^{-k}[1-(1-y_i^{-c})^{-k}]^{\alpha-1}\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}\ln(1-y_i^{-c})}{\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)}\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
L_\beta &= \frac{2n}{\beta} + \sum_{i=1}^{n}\left\{\ln\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\} + \ln\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}\right\} \\
&+ \sum_{i=1}^{n}\left\{\frac{\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\ln\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}}{1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}}\right\} \\
&+ \left\{\frac{\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}\ln\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}}{1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}}\right\} \\
&+ \sum_{i=1}^{n}\left\{\frac{\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\ln\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}}{\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\log\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)}\right\} \\
&+ \left\{\frac{\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta-1}\ln\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}}{\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\log\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)}\right\} \\
&+ \sum_{i=1}^{n}\left\{\left[1+(\psi_1+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)\right]\right\} \\
&\left[(\psi_2+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)\right]^{-1} \\
&\times\left[4(\psi_1+\psi_3)\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-2}\right] \\
&\times\left[-\log\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-2} \\
&\times\frac{\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\ln\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}}{\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)} \\
&+4(\psi_2+\psi_3)\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-2} \\
&\times\left[-\log\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-2} \\
&\times\left\{\left[\frac{\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\ln\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}}{\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)}\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
L_{\psi_1} &= \sum_{i=1}^{n}\left\{\left[1+(\psi_1+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)\right]\right\} \\
&+\left[(\psi_2+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)\right]^{-1} \\
&\times\left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right),
\end{aligned}
$$

$$
\begin{aligned}
L_{\psi_2} &= \sum_{i=1}^{n}\left\{\left[1+(\psi_1+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)\right]\right\} \\
&+\left[(\psi_2+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)\right]^{-1} \\
&\times\left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right),
\end{aligned}
$$

$$
\begin{aligned}
L_{\psi_3} &= \sum_{i=1}^{n}\left\{\left[1+(\psi_1+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)\right]\right\} \\
&+\left[(\psi_2+\psi_3)\left(1-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right)\right]^{-1} \\
&\times\left[2-2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-x_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right] \\
&-\left[2\left\{1+\left[-\log\left(1-\left\{1-\left[1-(1-y_i^{-c})^{-k}\right]^{\alpha}\right\}^{\beta}\right)\right]^{-1}\right\}^{-1}\right].
\end{aligned}
$$

Appendix C

Univariate case: Flood data 1

19.885, 20.940, 21.820, 23.700, 24.888, 25.460, 25.760, 26.720, 27.500, 28.100, 28.600, 30.200, 30.380, 31.500, 32.600, 32.680, 34.400, 35.347, 35.700, 38.100, 39.020, 39.200, 40.000, 40.400, 40.400, 42.250, 44.020, 44.730, 44.900, 46.300, 50.330, 51.442, 57.220, 58.700, 58.800, 61.200, 61.740, 65.440, 65.597, 66.000, 74.100, 75.800, 84.100, 106.600, 109.700, 121.970, 121.970, 185.560.

Univariate case: Earthquake data 2

1163, 3258, 323, 159, 756, 409, 501, 616, 398, 67, 896, 8592, 2039, 217, 9, 633, 461, 1821, 4863, 143, 182, 2117, 3709, 979

B-ivariate case: operational components of CPU

The bi-variate data set is as follows

Table 13: Operational lifetime of operational components of CPU

Sys	Processor lifetime	Memory lifetime	Sys	Processor lifetime	Memory lifetime	Sys	Processor lifetime	Memory lifetime
1	1.9292	3.9291	21	1.1739	3.3857	41	0.627	1.7289
2	3.6621	0.0026	22	1.3482	1.9705	42	0.7947	0.7947
3	3.9608	0.8323	23	3.0935	3.0935	43	0.5079	5.3535
4	2.3504	3.3364	24	2.1396	2.1548	44	2.5913	2.5913
5	1.0833	3.3059	25	1.3288	0.9689	45	2.5372	2.4923
6	2.8414	1.8438	26	0.1115	0.1115	46	1.1917	0.0801
7	0.3309	0.3309	27	0.8503	2.8578	47	1.5254	4.4088
8	2.9884	1.5961	28	0.1955	0.1955	48	1.0986	1.0986
9	0.5784	1.8795	29	0.4614	0.8584	49	1.0051	1.0051
10	0.552	0.552	30	3.3887	1.9796	50	1.364	1.364
11	1.9386	4.0043	31	0.1181	0.0884			
12	2.1	2.0513	32	5.0533	2.3238			
13	0.9867	0.9867	33	1.6465	2.0197			
14	0.1837	0.1837	34	0.9096	0.6214			
15	1.3989	4.1268	35	1.7494	2.3643			
16	2.3757	2.7953	36	0.1058	0.1058			
17	3.5202	1.4095	37	0.4593	0.4593			
18	2.3364	0.1624	38	0.9938	1.7689			
19	0.8584	1.9556	39	5.7561	0.3212			
20	4.3435	1.0001	40	6.295	1.0495			

References

[1] Azzalini. A, *A class of distributions which includes the normal ones*, Scandanavian Journal of Statistics, 12 (1985), pp. 171–178.

[2] A.W. Marshall and I. Olkin, *A new method for adding parameters to a family of distributions with application to the exponential and Weibull families*, Biometrika 84 (1997), pp. 641–652.

[3] R.C. Gupta, P.L. Gupta and R.D. Gupta, *Modeling failure time data by Lehman alternatives*, Commun. Stat. Theory Methods 27 (1998), pp. 887–904.

[4] J.U. Gleaton and J.D. Lynch, *Properties of generalized log-logistic families of lifetime distributions* J. Probab. Stat. Sci. 4 (2006), pp. 51–64.

[5] W.T. Shaw and I.R. Buckley, *The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map*, (2007), UCL discovery repository, http://discovery.ucl.ac.uk/id/eprint/643923.

[6] K. Zografos and N. Balakrishnan, *On families of beta- and generalized gamma generated distributions and associated inference*, Stat. Methodol. 6 (2009), pp. 344–362.

[7] M.M. Ristić and N. Balakrishnan, *The gamma-exponentiated exponential distribution*, J.Stat. Comput. Simul. 82 (2012), pp. 1191–1206.

[8] M. Bourguignon, R.B. Silva and G.M. Cordeiro, *The Weibull-G family of probability distributions*, J. Data. Sci. 12 (2014), pp. 53–68.

[9] N. Eugene, C. Lee and F. Famoye, *Beta-normal distribution and its applications*, Commun. Stat. Theory Methods 31 (2002), pp. 497–512.

[10] G.M. Cordeiro and M. de-Castro, *A new family of generalized distributions* J. Stat. Comput. Simul. 81 (2011), pp. 883–898.

[11] G.M. Cordeiro, E.M.M. Ortega and D.C.C. Cunha, *The exponentiated generalized class of distributions* J. Data. Sci. 11 (2013), pp. 1–27.

[12] C. Alexander, G.M. Cordeiro, E.M.M. Ortega and J.M. Sarabia, *Generalized beta-generated distributions* Comput. Stat. Data Anal. 56 (2012), pp. 1880–1897.

[13] M.H. Tahir and S. Nadarajah, *Parameter induction in continuous univariate distributions: Well-established G families*, An. Acad. Bras. Ciênc. 87 (2015), pp. 539–568.

[14] M.H. Tahir and G.M. Cordeiro, *Compounding of distributions: a survey and new generalized classes*, J. Stat. Dist. Applic. 3 (2016), Art. 13, doi:10.1186/s40488-016-0052-1.

[15] M. Alizadeh, M. C. Korkamaz, J. A. Almamy and A. A. E. Ahmed, Another odd log-logistic logarithmic class of continuous distributions. Istatistikciler Dergisi: Istatistik ve Aktuerya, 2018, 11.2: 55-72.

[16] M. C. Korkamaz, M. Alizadeh, H.M. Yousof, and G.G. Hamedani, The Topp-Leone generalized odd log-logistic family od distributions: Properties, characterizations and applications. Communications Faculty of Sciences University of Ankara Series A1: Mathematics and Statistics (2019), 69(2): 1303-5991.

[17] G.M. Cordeiro, E. Altun, M. C. Korkamaz, R. Pescim, A.Z. Afify and H.M. Yousof, The xgamma Family: Censored Regression Modelling and Applications. REVSTAT (2020), 18(5): 593612.

[18] G.G. Hamedani, M. C. Korkamaz; N.S. Butt and H.M. Yousof, The Type I Quasi Lambert Family: Properties, Characterizations and Different Estimation Methods. Mathematical and Statistical Science Faculty Research and Publications (2021), 106. https://epublications.marquette.edu/mathfac/106

[19] E. Altun, M. C. Korkamaz, M. El-Morshedy and M.S. Eliwa, A New Flexible Family of Continuous Distributions: The Additive Odd-G Family. Mathematics. 2021; 9(16):1837. https://doi.org/10.3390/math9161837

[20] Q. Ramzan, M. Amin, A. Elhassanein and M. Ikram, The extended generalized inverted Kumaraswamy Weibull distribution: Properties and applications, AIMS Mathematics 6 (9) (2021), pp. 9955-9980.

[21] Q. Ramzan, et al., On the extended generalized inverted Kumaraswamy distribution, Computational Intelligence and Neuroscience 2022 (2022), 1612959.

[22] R. A. R. Bantan et al., Statistical analysis of COVID-19 data: using a new univariate and bivariate statistical model, Journal of Function Spaces 2022 (2022), 2851352.

[23] A. Alzaatreh, F. Famoye and C. Lee, *A new method for generating families of continuous distributions*, Metron 71 (2013), pp. 63–79.

[24] H. Torabi and N.H. Montazari, *The logistic-uniform distribution and its application*, Commun. Stat. Simul. Comput. 43 (2014), pp. 2551–2569.

[25] M.H. Tahir, G.M. Cordeiro, M. Alizadeh, M. Mansoor and M. Zubair, *The logistic-X family of distributions and its applications*, Commun. Stat. Theory Methods 45 (2016), pp. 7326–7349.

[26] M. Mansoor, M.H. Tahir, G.M. Cordeiro, E.M.M. Ortega, A. Alzaatreh, *On analyzing non-monotone failure data*. To appear in REVSTAT Journal, 2021.

[27] S. Nasiru, P. N. Mwitaand O. Ngesa, *Exponentiated generalized transformed-transformer family of distributions*, J. of Statistical and Econometric Methods, 6(4) 2017,pp. 1–17.

[28] Alzaghal, A., Famoye, F. and Lee, C.(2013). Exponentiated T-X family of distributions with some applications. International Journal of Probability and Statistics 2, 31–49.

[29] G.S. Mudholker, D.K. Srivastava, *Exponentiated Weibull family for analyzing bathtub failure data*, IEEE Transactions of Reliability 42 (1993), pp. 299–302.

[30] R.D. Gupta and D. Kundu, *Exponentiated exponential family: An alternative to gamma and Weibull distributions*, Biometrical Journal 42 (2001), pp. 117–130.

[31] S. Nadarajah and S. Kotz, *The exponentiated type distributions*, Acta Applicandae Mathematica, 92 (2006a), pp.97–111.

[32] Apostol, T. M. (1974), Mathematical analysis, Addison-Wesley Pub. Co.

[33] Kleiber, C. and Kotz, S., *Statistical Size Distributions in Economics and Actuarial Sciences*, Hoboken, NJ, USA: Wiley-Interscience (2003); Application-stock returns.

[34] F. Domma and F. Condino, *The Beta-Dagum Distribution: Definition and Properties*, Communications in Statistics - Theory and Methods, 42:22 (2013), pp. 4070–4090.

[35] G.M. Cordeiro, E.M.M. Ortega, and S. Nadarajah, *The Kumaraswamy Weibull distribution with application to failure data*, J. Franklin Inst. 347 (2010), pp. 1399–1429.

[36] B.O. Oluyede, S. Huang and M. Pararai, *A new class of generalized Dagum distribution with application to income and lifetime data*, J. of Stat. and Econometric methods, 3 (2014), pp. 125–151.

[37] A. Y. Al-Saiari, S. A. Mousa and L. A. Baharith, *Marshall-Olkin extended Burr III distribution*, International Mathematical Forum, 11 (2016), pp. 631–642.

[38] A. Asgharzadeh, H.S. Bakouch and M. Habibi, *A generalized binomial exponential 2 distribution: modeling and applications to hydrologic events*, J. Appl. Statist. 44 (2017), pp. 2368–2387.

[39] Kus, C., *A new lifetime distribution*, Comput. Statist. Data Anal. 51 (2007), pp. 4497-4509., pp. 79–88.

[40] J. A.Darwish, L. I. Al turk, and M. Q. Shahbaz, Bivariate transmuted Burr distribution: Properties and applcations, Pak.j.stat.oper.res., 17 (1) (2021), pp. 15-24.

[41] M.Ganji, H. Bevraninand, and H. Golzar, A new method for generating a continuous bivariate distribution families, JIRSS, 17(1) (2018), pp. 109-129.

[42] H. Z. Muhammed, On a bivariate generalized inverted Kumaraswamy distribution, Physica A: Statistical Mechanics and its Applications, 553 (2020), article 124281.

[43] S. Mondal and D. Kundu, A bivariate inverse Weibull distribution and its applications in complementary risks models, Journal of Applied Statistics, 47 (6) (2021), pp. 1084–1108,

[44] J. A. Darwish and M. Q. Shahbaz, Bivariate transmuted Burr distribution: properties and applcations, Pakistan Journal of Statistics & Operation Research, 17 (1) (2021), pp. 15–24.

[45] M. V. O.Peres, J. A.Achcar and E. Z. Martinez, Bivariate lifetime models in presence of cure fraction: a comparative study with many different copula functions. Heliyon, 6 (2020), e03961.

[46] A. Elhassanein, On statistical properties of a ewn bivariate modified Lindley distribution with an application to financial data. Complexity, 2022 (2022), 2328831.

[47] A. Elhassanein, A new type I half-logistic inverse Weibull distribution with an application to the relief times data of patients receiving an analgesic. Journal of Medical Imaging and Health Informatics 12 (2) (2022), pp. 184-199.

[48] J.Navarro, Characterizations using the bivariate failure rate function. Statistics and Probability Letters, 78(12) (2010), p. 1349.

[49] T. S. Durrani and X. Zeng. (2007) "Copulas for bivariate probability distributions," Electronics Letters, 43 (4) (2007).

[50] R. P. Oliveira, J. A. Achcar, J. Mazucheli, and W. Bertoli, "A new class of bivariate Lindley distributions based on stress and shock models and some of their reliability properties,"Reliability Engineering and System Safety, 211 (2021), article 107528.

[51] M. Ragab and A. Elhassanein, A new bivariate extended generalized inverted Kumaraswamy Weibull distribution. Advances in Mathematical Physics, 2022 (2022), 1243018.