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The Almon Liu-Type M-Estimator for the Distributed Lag Models in the Presence of Multicollinearity and Outliers

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ABSTRACT

The Almon method is widely used for the estimation of the distributed lag models (DLM). The advantage of using the Almon technique lies in its capability to avoid some serious problems that may arise from the direct application of ordinary least squares (OLS). In the Almon technique, the OLS procedure is applied on transformed regressors, and these regressors correlate themselves leading to the problem of multicollinearity. Moreover, in the presence of outliers in the y-direction, the Almon estimator (AE) may become sensitive. The presence of multicollinearity and outliers jointly in the dataset can strongly distort the AE, leading to the unreliable estimation of the lagged coefficients. We propose the Almon Liu-type M-estimator (ALTME) to address the joint issue of multicollinearity and outliers in y-direction. To show that the proposed estimator has an advantage over the AE, the Almon M-estimator (AME), and the Almon ridge M-estimator (ARME), the Monte Carlo Simulation and two real-life numerical examples are given.

Keywords: Almon Estimator, Distributed Lag Models, Liu-type estimator, Ordinary Least Squares, Robust Estimation, Shrinkage parameter

1. Introduction

The distributed lag model (DLM) with finite lags is given as,

$$\begin{aligned} y_t &= \beta_0 x_t + \beta_1 x_{t-1} + \ldots + \beta_p x_{t-p} + u_t, t = p+1, \ldots, T \\ &= \sum_{i=0}^{p} \beta_i x_{t-i} + u_t \end{aligned} \tag{1}$$

where β_i is the unknown lag coefficients or lag weight, x_{t-i} is the t^{th} observations of explanatory variable at i^{th} lag period, y_t is the t^{th} observation on the response variable, u_t is unknown random error at time t which is normally distributed with mean zero and constant variance σ^2 i.e. $u_t \sim N(0, \sigma^2)$ and p is the lag length.

The model in Eq. (1) can be written in matrix notations as,

$$y = X\beta + u, \tag{2}$$

where

$$y = \begin{bmatrix} y_{p+1} \\ y_{p+2} \\ \vdots \\ y_T \end{bmatrix}, \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{bmatrix}, X = \begin{bmatrix} x_{p+1} & x_p & \cdots & x_1 \\ x_{p+1} & x_{p+1} & \cdots & x_2 \\ \vdots & \vdots & \vdots & \vdots \\ x_T & x_{T-1} & \cdots & x_{T-P} \end{bmatrix}, u = \begin{bmatrix} u_{p+1} \\ u_{p+2} \\ \vdots \\ u_T \end{bmatrix}$$

It is clear that y is a $(T-p)\times 1$ vector containing observations of response variable Y, β is a $(p+1)\times 1$ vector of unknown lag coefficients, X is a $(T-p)\times(p+1)$ matrix containing observations on explanatory variable and its p-period lag values, u is a $(T-p)\times 1$ vector of random errors.

The ordinary least squares (OLS) estimator (OLSE) of parameter vector β in model (2) is $\hat{\beta}_{OLS}=(X'X)^{-1}Xy$. The estimation of DLM by the method of OLS may have two serious problems:

a) It may not be possible to estimate parameters if the sample size is small and lag length is large enough because there may not be adequate degrees of freedom to perform traditional tests of significance.
b) The successive (lag) values may be highly correlated, leading to the problem of multicollinearity, which results in imprecise estimation of parameters.

To tackle the aforementioned problems, Almon (1965) proposed a new approach for estimation of the DLM which has gained much popularity among the practitioners. Almon (1965) assumed that β_is can be well approximated by a polynomial of suitable degree r in i which is less than p (lag length) i.e.

$$\beta_i=\alpha_0+\alpha_1 i+\alpha_2 i^2+\ldots+\alpha_r i^r\,; \qquad i=0,1,2,\cdots,p \quad and \quad p\geq r\geq 0 \tag{3}$$

By substituting Eq. (3) in Eq. (1), one can estimate α_i by the usual OLS procedure and then the estimates of β_i can be obtained by Eq. (3). In matrix notations, Eq. (3) can be written as

$$\beta=R\alpha, \tag{4}$$

where β is as defined earlier and

$$R=\begin{bmatrix}1 & 0 & 0 & \ldots & 0\\ 1 & 1 & 1 & \cdots & 1\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & p & p^2 & \cdots & p^r\end{bmatrix}, \qquad \alpha=\begin{bmatrix}\alpha_0\\ \alpha_1\\ \vdots\\ \alpha_r\end{bmatrix} \tag{5}$$

are $(p+1)\times(r+1)$ matrix and $(r+1)\times 1$ vector, respectively. After substituting Eq. (4) in Eq. (2), we get

$$\begin{aligned} y &= XR\alpha+u\\ &= Z\alpha+u, \qquad where\ \ Z=XR \end{aligned} \tag{6}$$

The OLSE of α is,

$$\hat{\alpha}=(Z'Z)^{-1}Z'y \tag{7}$$

The parameter vector β can be estimated as,

$$\hat{\beta}=R\hat{\alpha}. \tag{8}$$

The estimator $\hat{\beta}$ is known as the Almon estimator (AE) of β. The AE is the best linear unbiased estimator of β, i.e., $E(\hat{\beta})=\beta$, under the assumption that $\beta=R\alpha$ (Judge et al. 1980). A major advantage of the Almon technique is that if a distributed lag is assumed to lie on a polynomial of a specified degree, then the distributed lag can be estimated by standard linear regression methods (Fair and Jaffee, 1971).

Moreover, the OLSE in Eq. (7) depends on the characteristics of $Z'Z$. If any two columns of the matrix $Z'Z$ are linearly dependent and some of the eigenvalues are very small, then the design matrix is said to be ill-conditioned which is the indication of presence of multicollinearity. In such situation, the variance of OLSE inflates, making it difficult to find statistically significant results.

Hoerl and Kennard (1970) proposed the ridge regression estimator (RRE) to handle the issue of multicollinearity in the linear regression model, by adding a small constant in the diagonal of the design matrix. Following Hoerl and Kennard (1970), the Almon ridge estimator (ARE) is proposed in the literature to tackle the issue of multicollinearity in the DLM see, for examples, Güler et al. (2017) and Lukman & Kibria (2021). The ARE is defined as

$$\begin{aligned}\hat{\alpha}(k) &= (Z'Z+kI)^{-1}Z'y\\ &= (Z'Z+kI)^{-1}Z'Z\hat{\alpha},\end{aligned}$$

where $k>0$ and is called the shrinkage parameter.

Moreover, the presence of outliers in the dataset may also adversely affect OLSE. The OLSE may inaccurately depict the underlying relationship between variables when errors are heavy tailed as well as the distribution of errors is non-normal. Further, the presence of outliers in the dataset can dramatically change the OLS estimates, resulting to significant changes not only in the magnitude of OLS estimate as well as in the direction of estimated coefficients. There can be two possible solutions to address the issue of outliers. First is to remove the outlier from dataset and the second is to use some robust estimation methods. Several robust estimators are available in the literature to address the issue of outliers in linear regression method. Among the available robust estimators, the M-estimator has gained widespread popularity. The M-estimation technique is widely used by the researcher when dealing with a dataset which is contaminated with outliers. Thus, when dataset is contaminated with outliers, using OLSE combined with the Almon technique can cast doubt on reliability of the estimates of lag coefficient. Majid and Aslam (2023) proposed the Almon M-estimator (AME) to address the problem of outliers in DLM. The AME for the parameter vector β is defined as

$$\hat{\beta}_M = R\hat{\alpha}_M \tag{9}$$

where $\hat{\alpha}_M$ is the M-estimator of α in Model (6).

The joint problem of multicollinearity and outliers may also arise in the estimation of the DLM. To tackle the combined problem of multicollinearity and outliers in the DLM, Majid et al. (2024) proposed the Almon ridge M-estimator (ARME). The ARME is obtained by shrinking the $\hat{\alpha}_M$ instead of $\hat{\alpha}$. The ARME of Model (6) can be defined as:

$$\hat{\alpha}_M(k) = (Z'Z + kI)Z'Z\hat{\alpha}_M$$

Majid et al. (2024) showed the superiority of their proposed estimator (ARME) over the AE and the AME by providing two real life dataset and the Monte Carlo Simulation.

Rest of the article is organized as follows: In section 2, we propose the new robust biased estimator. The MSE of the proposed estimator is presented in section 3. Moreover, the MSE comparison of proposed estimator with the AME and the ARME is also presented in the same section. Section 4 shows the derivation of the shrinkage parameters k and d. In section 5, the Monte Carlo simulation scheme is given along with the results. The application of proposed estimator for two real life data sets is also given in section 6. Finally, Section 7 concludes the article.

2. The Proposed Estimator

The RRE is a popular alternative approach to the OLSE for handling the issue of multicollinearity. The addition of the shrinkage parameter k in the diagonals of design matrix $(Z'Z)$ improves the condition number. However, the condition number of $Z'Z + kI$ is clearly a decreasing function of k, and in order to keep the condition number of $Z'Z + kI$ under control, k should be selected large. It is possible that the small k value is insufficient to address the ill-conditioning problem. In such situation, the RRE might be unstable because $Z'Z + kI$ is still ill-conditioned (Liu, 2003). To cope with this issue with the RRE, the Liu-type estimator is available in literature, see, Liu (2003). Following Liu (2003), the Almon Liu-type estimator for Model (6) can be defined as

$$\hat{\alpha}(k,d) = (Z'Z + kI)^{-1}(Z'y - d\alpha^*),$$

where $k > 0, -\infty < d < \infty$ and α^* be any estimator of α. Following Liu (2003), two different alternatives for the α^* are $\alpha^* = \hat{\alpha}$ and $\alpha^* = \hat{\alpha}(k)$. When $\alpha^* = \hat{\alpha}$, then $\hat{\alpha}(k,d)$ is given as

$$\hat{\alpha}(k,d) = (Z'Z + kI)^{-1}(Z'Z - dI)\hat{\alpha} = W_1\hat{\alpha},$$

and when $\alpha^* = \hat{\alpha}(k)$, then $\tilde{\alpha}(k,d)$ is given as

$$\tilde{\alpha}(k,d) = (Z'Z + kI)^{-1}(Z'Z - d(Z'Z + kI)^{-1}Z'Z)\,\hat{\alpha} = W_2\hat{\alpha}.$$

Moreover, the presence of outliers in y-direction may have serious impact on the estimates, as these are obtained by shrinking the OLSE. Ertaş et al. (2017) proposed the robust Liu-type estimator to address the joint problem of multicollinearity and outliers in linear regression model. Adopting the lines of Ertaş et al. (2017), we propose the Almon Liu-type M-estimator (ALTME) to tackle the joint issue of multicollinearity and outliers in the DLM. The ALTME is obtained by shrinking the $\hat{\alpha}_M$ instead of $\hat{\alpha}$ using the matrices W_1 and W_2. The ALTME can be defined as

$$\hat{\alpha}_M(k,d) = W_1\hat{\alpha}_M,$$

and

$$\tilde{\alpha}_M(k,d) = W_2\hat{\alpha}_M.$$

In this case, the parameter vector β can be estimated as,

$$\hat{\beta}_{ALTM} = R\hat{\alpha}_M(k,d)$$

and

$$\tilde{\beta}_{ALTM} = R\tilde{\alpha}_M(k,d).$$

Consider the canonical form of the model in Eq. (6)

$$y = W\gamma + u$$

where W=ZQ, $\gamma = Q'\alpha$ and Q is an orthogonal matrix whose columns consist of eigenvectors of $Z'Z$. Then $W'W = QZ'ZQ = \Lambda = diag\{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{r+1}\}$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{r+1}$be the ordered eigenvalues. The estimators can be defined in canonical form as follows:

$$\hat{\gamma}(k) = (\Lambda + kI)^{-1}\Lambda\hat{\gamma},$$
$$\hat{\gamma}_M(k) = (\Lambda + kI)^{-1}\Lambda\hat{\gamma}_M,$$
$$\hat{\gamma}_M(k,d) = (\Lambda + kI)^{-1}(\Lambda - dI)\hat{\gamma}_M,$$
$$\hat{\gamma}_M(k,d) = W_1^*\hat{\gamma}_M, \qquad k > 0, -\infty < d < \infty$$
$$\tilde{\gamma}_M(k,d) = (\Lambda + \mathrm{k}\mathrm{I})^{-1}(\Lambda - d(\Lambda + kI)^{-1}\Lambda)\hat{\gamma}_M,$$
$$\tilde{\gamma}_M(k,d) = W_2^*\hat{\gamma}_M, \quad k > 0, -\infty < d < \infty$$

where $W_1^* = (\Lambda + kI)^{-1}(\Lambda - dI)$ and $W_2^* = (\Lambda + \mathrm{k}I)^{-1}(\Lambda - d(\Lambda + kI)^{-1}\Lambda)$.

3. Performance of the Proposed ALTME using MSE criterion

MSE of the estimator $\hat{\alpha}$ of parameter α is defined as

$$MSE(\hat{\alpha}) = E(\hat{\alpha} - \alpha)'(\hat{\alpha} - \alpha) = tr[Cov(\hat{\alpha}) + (Bias(\hat{\alpha}))'Bias(\hat{\alpha})]$$

where

$$Cov(\hat{\alpha}) = E(\hat{\alpha} - E(\hat{\alpha}))'(\hat{\alpha} - E(\hat{\alpha}))$$

and

$$Bias(\hat{\alpha}) = E(\hat{\alpha}) - \alpha$$

Note that $\hat{\gamma}$ is any estimator of γ having a corresponding relation $\hat{\alpha} = Q'\hat{\gamma}$ such that $MSE(\hat{\alpha}) = MSE(\hat{\gamma})$. Thus, we may restrict our discussion to the canonical form. The MSE expressions of the estimators under study are given as:

$$MSE\ (\hat{\gamma}) = \sigma^2 \sum_{j=1}^{(r+1)} \frac{1}{\lambda_j} \tag{10}$$

$$MSE(\hat{\gamma}_M) = \sum_{j=1}^{(r+1)} \Omega_{jj} \tag{11}$$

$$MSE(\hat{\gamma}_M(k)) = \sum_{j=1}^{(r+1)} \frac{\lambda_j^2}{(\lambda_j + k)^2}\Omega_{jj} + k^2 \sum_{j=1}^{(r+1)} \frac{\hat{\gamma}_j^2}{(\lambda_j + k)^2} \tag{12}$$

$$MSE(\hat{\gamma}_M(k,d)) = \sum_{j=1}^{(r+1)} \frac{(d - \lambda_j)^2}{(\lambda_j + k)^2}\Omega_{jj} + \sum_{j=1}^{(r+1)} \frac{(d + k)^2\gamma_j^2}{(\lambda_j + k)^2} \tag{13}$$

$$MSE(\tilde{\gamma}_M(k,d)) = \sum_{j=1}^{(r+1)} \frac{\left(\lambda_j - \frac{d\lambda_j}{\lambda_j + k}\right)^2}{(\lambda_j + k)^2}\Omega_{jj} + \sum_{j=1}^{(r+1)} \frac{(k\lambda_j + k^2 + d\lambda_j)^2\gamma_j^2}{(\lambda_j + k)^4} \tag{14}$$

where $\Omega = Cov(\hat{\gamma}_M)$.

3.1 Comparison between $\hat{\gamma}_M(k)$, $\hat{\gamma}_M(k,d)$, and $\tilde{\gamma}_M(k,d)$

Before stating the existing theorem, we assume that the following conditions hold:

- ψ is skew symmetric and non-decreasing.
- The errors are systematic.
- $\Omega = Cov(\gamma_M)$ is finite.

Theorem 3.1. Assuming that the above conditions hold and Ω_{jj} $(j = 1,2,\ldots,r+1)$ are the diagonal elements of Ω. If $\Omega_{jj} < \sigma^2\lambda_j^{-1}$ then $MSE(\hat{\gamma}_M(k,d)) < MSE(\hat{\gamma}_M(k))$ and $MSE\left(\tilde{\gamma}_M(k,d)\right) < MSE(\hat{\gamma}_M(k))$ for every $k > 0$ and $-\infty < d < \infty$.

Proof. The MSE expression of $\hat{\gamma}_M(k)$, $\hat{\gamma}_M(k,d)$, and $\tilde{\gamma}_M(k,d)$ respectively,

$$MSE\left(\hat{\gamma}_M(k)\right) = \sum_{j=1}^{(r+1)} \frac{\lambda_j^2}{\left(\lambda_j + k\right)^2}\Omega_{jj} + k^2 \sum_{j=1}^{(r+1)} \frac{\hat{\gamma}_j^2}{\left(\lambda_j + k\right)^2}$$

The difference between $MSE\left(\hat{\gamma}_M(k,d)\right) - MSE\left(\hat{\gamma}_M(k)\right)$ is denoted by D and difference between $MSE\left(\tilde{\gamma}_M(k,d)\right) - MSE\left(\hat{\gamma}_M(k)\right)$ is denoted by D^*. After some algebraic manipulation we get,

$$D = d\sum_{j=1}^{(r+1)} \frac{\Omega_{jj} + \gamma_j^2}{(\lambda + k)^2} - 2\sum_{j=1}^{(r+1)} \frac{\lambda\,\Omega_{jj} - k\gamma^2}{(\lambda + k)^2}$$

and

$$D^* = \sum_{j=1}^{(r+1)} \frac{(\lambda d)^2(\Omega_{jj} + \gamma_j^2)}{(\lambda + k)^4} - 2d\sum_{j=1}^{(r+1)} \frac{(\lambda^2 - k\lambda)(\Omega_{jj} - \gamma^2)}{(\lambda + k)^3}$$

If k and d lies between 0 and 1 then our proposed estimator $\hat{\gamma}_M(k,d)$ and $\tilde{\gamma}_M(k,d)$ will be superior.

4. Estimation of the Shrinkage Parameters

The optimum value of biasing parameter d is obtained by minimizing $MSE(\hat{\gamma}_M(k,d))$ and $MSE(\tilde{\gamma}_M(k,d))$. By differentiating Eq. (13) and Eq. (14) with respect to d, equating the derivatives to zero and after some algebraic manipulation we get

$$d_{opt} = \frac{\sum_{j=1}^{r+1}(\lambda_j\Omega_{jj} - k\gamma_j^2)/\left(\lambda_j + k\right)^2}{\sum_{j=1}^{r+1}(\Omega_{jj} + \gamma_j^2)/\left(\lambda_j + k\right)^2} \quad (15)$$

$$d_{opt}^* = \frac{\sum_{j=1}^{r+1}\frac{\lambda_j^2\Omega_{jj} - k\,\gamma_j^2\lambda_j}{\left(\lambda_j + k\right)^3}}{\sum_{j=1}^{r+1}\frac{\lambda_j^2\left(\Omega_{jj} + \gamma_j^2\right)}{\left(\lambda_j + k\right)^4}} \quad (16)$$

where, d_{opt} and d_{opt}^* depend on the unknown Ω_{jj} and γ_j. The estimates of d_{opt} and d_{opt}^* are obtained by substituting unbiased estimates of γ_j and Ω_{jj} in Eq. (15) and (16). We assume that $\hat{\gamma}_M$ is normally distributed with mean γ and covariance matrix $A^2\,\Lambda^{-1}$. This supposition holds approximately since $(\hat{\gamma}_M - \gamma) \xrightarrow{d} N(0, A^2\Lambda^{-1})$, where

$$A^2 = s_0^2 E\frac{\left[\psi^2\left(\frac{\varepsilon}{s_0}\right)\right]}{\left[E\psi'\left(\frac{\varepsilon}{s_0}\right)\right]^2}$$

with scale estimate s_0 (Huber 1981). So, the unbiased estimate of γ_j is $\hat{\gamma}_{Mj}$ and the unbiased estimator of Ω_{jj} is asymptotically $\hat{A}^2/\lambda_j$, where

$$\hat{A}^2 = \frac{s^2(n-p)^{-1}\sum_{i=1}^{n}\psi^2(\tilde{u}_i/s)}{\left\{n^{-1}\sum_{i=1}^{n}\psi'(\tilde{u}_i/s)\right\}^2}$$

with $\tilde{u} = (y - W\hat{\gamma}_M)$. By replacing γ_j and Ω_{jj} by their unbiased estimates in Eq. (15) and (16), we get the optimal estimator of d_{opt} and d_{opt}^* as,

$$\hat{d} = \frac{\sum_{j=1}^{(r+1)}(\hat{A}^2 - k\,\hat{\gamma}_M^2(k))/\left(\lambda_j + k\right)^2}{\sum_{j=1}^{(r+1)}(\hat{A}^2 + \lambda_j\,\hat{\gamma}_M^2(k))/\lambda_j\left(\lambda_j + k\right)^2}$$

$$\tilde{d} = \frac{\sum_{j=1}^{(r+1)}(\hat{A}^2\lambda_i - k\hat{\gamma}_M^2(k)\lambda_i)/\left(\lambda_j + k\right)^3}{\sum_{j=1}^{(r+1)}(\hat{A}^2\lambda_i + \lambda_j\,\hat{\gamma}_M^2(k)\lambda_i^2)/\left(\lambda_j + k\right)^4}$$

where k can be estimated as

$$\hat{k} = \frac{\lambda_1 - 100 * \lambda_p}{99}\,(Liu, 2003)$$

and

$$\hat{k}_{AM} = \frac{1}{p}\sum_{j=1}^{(r+1)} \frac{\hat{\sigma}^2}{\alpha_j^2} (Kibria, 2003)$$

In the literature, numerous methods have been proposed for estimating the ridge parameter k. However, in this study we focused on the above-mentioned two estimators introduced by Liu (2003) and Kibria (2003). Prior simulation studies, as well as later extensions of these works, have shown that Liu's estimator achieves a favourable bias–variance trade-off, particularly under moderate to severe multicollinearity, while Kibria's estimator consistently outperforms classical choices such as the Hoerl–Kennard (1970) estimator across different correlation structures and sample sizes. Moreover, both approaches are computationally simple, requiring no iterative optimization or cross-validation, which makes them especially practical for large datasets and well suited to the applied orientation of our study.

5. The Monte Carlo Simulation Study

The primary objective of this study is to introduce novel, efficient, and robust estimator for the DLM and to compare the performance of our proposed estimator with the already available estimators. Following Frost (1975), Özbay and Kaçıranlar (2017), Majid and Aslam (2023) and Majid et al. (2024), the observations are generated by the following model:

$$y_t = \beta_0 + \beta_1 x_{t-1} + \ldots + \beta_p x_{t-p} + u_t \quad (17)$$

$$x_1 = v_1 \quad (18)$$

$$x_t = \lambda x_{t-1} + (1-\lambda^2)^{\frac{1}{2}} v_t \quad (19)$$

where u_t are independently and normally distributed with mean zero and variance σ^2. The variate v_t are independent and normal with a zero mean and a unit standard deviation. The coefficient vector β is chosen to be the normalized eigenvector corresponding to the largest eigenvalue of $X'X$, such that $\beta'\beta = 1$. The lag length p is taken 10 and the degree of polynomial $r = 2$ is used. The coefficient λ indicates the expected correlation between successive values of x_t and is set to be 0.90, 0.95, 0.99 and 0.999. We consider the sample sizes $T = 50, 100, 150,$ and 200 in this study. For sample size $T = 50$, the data are generated by drawing values of v_t for $t = 1, 2, 3, \ldots, 50$ and u_t for $t = 11, 2, 3, \ldots, 50$. The values of x_t are generated using Eq. (18) and (19). For T = 50, each replication allows for the regression with T = 40 observations ($t = 11, 2, 3, \ldots, 50$) to be run with lags up to 10 periods. The observations y_t are generated by Eq. (17). To investigate the effect of the number of outlier cases on the estimators, we consider three different scenarios, i.e. one outlier, two outliers, and four outliers in the datasets. To introduce outliers in the dependent variable, we follow Ertaş (2017), Majid and Aslam (2023) and Majid et al. (2024). For the case of one outlier, values of the dependent variable y_t are generated by Eq. (17) and we introduced an outlier in the y-direction by altering the twelfth observation as $y_{12}^* = y_{12} + 20\hat{\sigma}$. For two outliers' case, the twelfth and sixteenth observation are modified as $y_{12}^* = y_{12} + 20\hat{\sigma}$ and $y_{16}^* = y_{16} - 25\hat{\sigma}$. Similarly, for the four outliers' case $y_{15}^* = y_{15} + 20\hat{\sigma}$, $y_{16}^* = y_{16} - 25\hat{\sigma}$, $y_{23}^* = y_{23} + 5\hat{\sigma}$ and $y_{29}^* = y_{29} - 15\hat{\sigma}$ are altered. The observations of x_t and y_t for large samples are generated in the same manner as discussed above. After constructing the matrix X and Z is obtained as $Z = XR$. Once the matrices X and Z are constructed, they are kept fixed throughout the simulations.

All the computations are carried out using programming routines in statistical package R 4.2.0. The number of the Monte Carlo replications is taken to be 5000. For the computation of M-estimator, the *rlm*( ) procedure is used with Huber's (1981) objective function.

5.1 Criteria for Numerical Evaluation

Performance of proposed estimators is compared on the basis of two evaluation criteria i.e., the MSE and the mean absolute error (MAE). There are several studies in the literature which have used these measures to evaluate the performance of estimators, see, e.g. Kibria (2003, 2022), Månsson, Shukur, Kibria (2010), Qasim, Amin, and Omer (2019), Majid and Aslam (2023) and Majid et al. (2024). For any particular estimator $\hat{\beta}$ of β, MSE and MAE are given by

$$MSE(\hat{\beta}) = E(\hat{\beta} - \beta)'(\hat{\beta} - \beta),$$

$$MAE(\hat{\beta}) = E|\hat{\beta} - \beta|$$

However, for evaluation purposes, the estimated MSE and MAE for any estimator $\hat{\beta}$ of β can be computed as

$$EMSE(\hat{\beta}) = \frac{\sum_{i=1}^{M}(\hat{\beta}_{(i)} - \beta)'(\hat{\beta}_{(i)} - \beta)}{R} \quad (20)$$

and

$$EMAE(\hat{\beta}) = \frac{\sum_{i=1}^{M}|\hat{\beta}_{(i)} - \beta|}{R},$$

where R is the number of repetitions in a simulation and $\hat{\beta}_{(i)}$ is the estimated value of β in the i^{th} repetition.

Table 1. Estimated MSE values of the estimators with one outlier.

T	λ	$\hat{\alpha}$	$\hat{\alpha}_M$	$\hat{\alpha}_M(k)$	$\hat{\alpha}_M(k,d)$		$\tilde{\alpha}_M(k,d)$	
					$(\hat{k},\hat{d})$	$(\hat{k}_{AM},\hat{d})$	$(\hat{k},\tilde{d})$	$(\hat{k}_{AM},\tilde{d})$
				$\sigma = 1$				
50	0.9	0.2786	0.0814	0.0556	0.0479	**0.0474**	0.0790	0.1076
	0.95	0.5812	0.1544	0.0863	0.0728	**0.0666**	0.1666	0.2679
	0.99	3.4048	0.7530	0.3683	0.2651	**0.2331**	0.2603	0.4683
	0.999	36.2039	7.6382	3.5193	2.6742	2.1978	**0.9702**	12.3973
100	0.9	0.1207	0.0563	**0.0473**	0.0622	0.0624	0.0646	0.0646
	0.95	0.2561	0.0879	0.0694	0.0705	**0.0692**	0.1219	0.1369
	0.99	1.0448	0.3603	0.1785	0.1125	**0.1114**	0.1500	0.1980
	0.999	10.9566	3.7234	1.7179	1.1410	0.9794	**0.6024**	6.1175
150	0.9	0.0677	0.0281	**0.0264**	0.0325	0.0332	0.0363	0.0363
	0.95	0.1108	0.0490	0.0309	**0.0260**	0.0286	0.0428	0.0436
	0.99	0.5244	0.2338	0.1146	**0.0692**	0.0756	0.0998	0.1155
	0.999	5.0000	2.3252	1.0318	0.6367	0.5633	**0.4156**	1.1305
200	0.9	0.0348	0.0182	**0.0119**	0.0164	0.0161	0.0275	0.0275
	0.95	0.0599	0.0324	**0.0172**	0.0213	0.0211	0.0289	0.0289
	0.99	0.2437	0.1521	0.0677	**0.0471**	0.0496	0.0548	0.0544
	0.999	2.2983	1.5120	0.5883	0.2096	0.2108	**0.2049**	0.2213
				$\sigma = 5$				
50	0.9	6.9462	1.9945	1.0042	0.9541	0.8771	**0.5646**	62.4953
	0.95	14.3284	3.7329	1.7825	1.5654	1.3678	**0.6984**	55.9869
	0.99	84.0464	18.1174	8.0677	5.8539	4.6899	**1.6919**	553.8962
	0.999	894.0156	190.6777	86.6492	66.6574	54.5175	**16.7907**	1150.9055
100	0.9	2.8974	1.1144	0.6383	0.5714	0.5467	**0.4584**	2.4639
	0.95	6.1649	2.0783	1.0801	0.9335	0.8610	**0.5348**	2.4706
	0.99	26.0419	8.8774	3.9936	2.4844	2.0711	**0.8939**	113.4854
	0.999	273.2148	92.4560	41.6237	27.4309	22.8458	**6.7637**	205.2232
150	0.9	1.6998	0.7209	0.3903	0.3172	0.2982	**0.2883**	0.5487
	0.95	2.7929	1.2551	0.5902	0.4161	0.3760	**0.3118**	0.5531
	0.99	13.0054	5.7433	2.4988	1.5507	1.3293	**0.6759**	387.3892
	0.999	124.0316	59.0254	25.6465	15.6738	13.1733	**3.7798**	115.7014
200	0.9	0.8531	0.4526	0.2198	**0.1406**	0.1410	0.1572	0.1804
	0.95	1.5087	0.8298	0.3558	0.1827	0.1827	**0.1818**	0.2064
	0.99	5.8926	3.7342	1.4442	0.4990	0.4560	**0.3100**	0.4135
	0.999	56.8211	37.7519	14.3666	4.5243	3.9350	**1.1483**	9.1825
				$\sigma = 10$				
50	0.9	27.8363	8.0714	3.9516	3.8096	3.5100	**1.2996**	308.8032
	0.95	57.3888	14.9830	7.0086	6.1902	5.4046	**1.8051**	589.7519

T	λ	$\hat{\alpha}$	$\hat{\alpha}_M$	$\hat{\alpha}_M(k)$	$\hat{\alpha}_M(k,d)$		$\tilde{\alpha}_M(k,d)$	
					$(\hat{k},\hat{d})$	$(\hat{k}_{AM},\hat{d})$	$(\hat{k},\tilde{d})$	$(\hat{k}_{AM},\tilde{d})$
	0.99	340.4642	75.5961	33.8935	24.5989	19.7418	**6.0649**	562.6892
	0.999	3559.6071	744.3623	332.7772	259.3634	209.5316	**64.1146**	13812.4759
100	0.9	11.4349	4.4497	2.3309	2.1384	2.0246	**0.9758**	138.1409
	0.95	24.5515	8.1118	4.0437	3.4926	3.2107	**1.2969**	50.8823
	0.99	105.4165	36.0232	16.1334	10.0184	8.3115	**2.6877**	126.2393
	0.999	1095.4913	371.2489	168.0943	111.2161	93.8059	**27.1842**	12038.1222
150	0.9	6.6803	2.7772	1.3891	1.1669	1.0782	**0.6330**	35.9213
	0.95	11.3020	4.9763	2.2759	1.6568	1.4695	**0.7201**	167.8772
	0.99	53.2151	23.7342	10.2631	6.1753	5.1908	**1.7045**	29.8802
	0.999	492.7807	233.9318	101.0480	63.4799	53.3575	**13.8523**	6860.9084
200	0.9	3.4689	1.8245	0.8174	0.5327	0.4777	**0.3596**	0.5931
	0.95	6.1016	3.3623	1.3937	0.7048	0.6258	**0.4007**	0.8112
	0.99	23.5257	15.0847	5.7702	1.9803	1.7203	**0.6565**	4.4625
	0.999	230.7401	152.6051	58.0441	18.3920	15.8693	**3.5167**	48.9708

Table 2. Estimated MSE values of the estimators with two outliers.

T	λ	$\hat{\alpha}$	$\hat{\alpha}_M$	$\hat{\alpha}_M(k)$	$\hat{\alpha}_M(k,d)$		$\tilde{\alpha}_M(k,d)$	
					$(\hat{k},\hat{d})$	$(\hat{k}_{AM},\hat{d})$	$(\hat{k},\tilde{d})$	$(\hat{k}_{AM},\tilde{d})$
				$\sigma = 1$				
50	0.9	1.5449	0.0895	0.0662	**0.0566**	0.0577	0.0950	0.1297
	0.95	3.9290	0.1775	0.1094	0.0917	**0.0857**	0.1794	0.2642
	0.99	16.6785	0.8246	0.4366	0.3147	**0.2778**	0.2853	0.4965
	0.999	191.2412	8.5708	4.3109	3.0785	2.6053	**1.0566**	7.9006
100	0.9	0.7100	**0.0505**	0.0555	0.0697	0.0695	0.0737	0.0725
	0.95	1.7316	0.0957	0.0775	0.0789	**0.0764**	0.1147	0.1251
	0.99	3.1115	0.3692	0.1833	0.1146	**0.1131**	0.1572	0.2014
	0.999	34.0374	3.7581	1.7510	1.1452	0.9825	**0.5954**	4.4660
150	0.9	0.2297	0.0299	**0.0259**	0.0318	0.0325	0.0327	0.0330
	0.95	0.3445	0.0523	0.0331	**0.0284**	0.0310	0.0408	0.0422
	0.99	1.3329	0.2414	0.1241	**0.0791**	0.0850	0.1111	0.1312
	0.999	12.8293	2.3635	1.0594	0.6566	0.5776	**0.4298**	1.3945
200	0.9	0.0563	0.0186	**0.0120**	0.0168	0.0165	0.0247	0.0248
	0.95	0.0923	0.0324	**0.0176**	0.0227	0.0225	0.0272	0.0272
	0.99	0.3499	0.1618	0.0727	**0.0512**	0.0539	0.0558	0.0550
	0.999	3.4482	1.5470	0.6106	0.2291	0.2298	**0.2266**	0.2411
				$\sigma = 5$				
50	0.9	38.8530	2.2531	1.2872	1.1778	1.1116	**0.7457**	84.7729
	0.95	97.7395	4.2353	2.2605	1.9457	1.7387	**0.8586**	5119.9068
	0.99	417.4419	20.2771	9.8982	6.8455	5.6550	**1.8494**	76.3222
	0.999	4778.9264	213.2991	106.3122	76.5857	64.3844	**19.1894**	528.1793
100	0.9	17.8260	1.2087	0.7012	0.6544	0.6211	**0.4681**	1.2157
	0.95	43.4391	2.3072	1.2364	1.0730	0.9912	**0.5939**	1.9044
	0.99	76.9051	9.4367	4.2798	2.5699	2.1599	**0.8513**	1608.7036
	0.999	845.5111	92.9126	41.9697	27.3589	22.6645	**6.4059**	548.5215
150	0.9	5.6414	0.7361	0.4121	0.3290	**0.3130**	0.3330	1.1774
	0.95	8.6167	1.2876	0.6176	0.4316	0.3943	**0.3489**	0.7627

T	λ	$\hat{\alpha}$	$\hat{\alpha}_M$	$\hat{\alpha}_M(k)$	$\hat{\alpha}_M(k,d)$		$\tilde{\alpha}_M(k,d)$	
					$(\hat{k},\hat{d})$	$(\hat{k}_{AM},\hat{d})$	$(\hat{k},\tilde{d})$	$(\hat{k}_{AM},\tilde{d})$
	0.99	32.8924	5.9122	2.5749	1.5669	1.3371	**0.6797**	14.1930
	0.999	325.7394	59.8238	25.5660	15.4288	12.9548	**3.8219**	148.2516
200	0.9	1.4255	0.4701	0.2306	**0.1435**	0.1439	0.1613	0.1909
	0.95	2.3832	0.8327	0.3590	0.1925	**0.1916**	0.1983	0.2305
	0.99	8.4744	3.8085	1.4620	0.5194	0.4751	**0.3226**	0.4248
	0.999	85.9895	38.1848	14.6321	4.6236	4.0395	**1.1654**	31.2606
				$\sigma = 10$				
50	0.9	155.5842	8.9995	4.9759	4.6097	4.3347	**1.8371**	906.6408
	0.95	392.5158	17.4175	9.2106	7.8422	7.0093	**2.5482**	168.9487
	0.99	1678.4621	82.4692	40.5173	28.1358	23.3008	**6.6000**	230.2423
	0.999	19114.4330	850.2534	422.2555	309.3556	259.3823	**74.1511**	2894.2534
100	0.9	71.4254	4.9581	2.7095	2.5050	2.3749	**1.1615**	11.2686
	0.95	172.6783	9.2866	4.8304	4.1376	3.8285	**1.6435**	44.6398
	0.99	308.3430	37.2704	16.8268	10.0768	8.3837	**2.5964**	4188.1327
	0.999	3407.4250	374.8622	168.7488	108.6382	91.5157	**26.4385**	693.1398
150	0.9	22.4923	2.9124	1.4990	1.2633	1.1800	**0.7091**	17.8189
	0.95	34.3236	5.0552	2.3683	1.7332	1.5461	**0.7820**	33618.2339
	0.99	131.1258	24.0759	10.4737	6.2990	5.3293	**1.7438**	41.7041
	0.999	1300.5335	240.9435	103.1525	62.0528	52.1211	**13.8661**	772.8064
200	0.9	5.7081	1.8591	0.8356	0.5394	0.4839	**0.3689**	0.7009
	0.95	9.4066	3.2631	1.3504	0.7186	0.6340	**0.4186**	0.9244
	0.99	34.4780	15.4388	5.8976	1.9736	1.7153	**0.6707**	5.2334
	0.999	341.4201	153.7778	58.4108	18.7807	16.1228	**3.6681**	74.0650

Table 3. Estimated MSE *values* of the estimators with four outliers.

T	λ	$\hat{\alpha}$	$\hat{\alpha}_M$	$\hat{\alpha}_M(k)$	$\hat{\alpha}_M(k,d)$		$\tilde{\alpha}_M(k,d)$	
					$(\hat{k},\hat{d})$	$(\hat{k}_{AM},\hat{d})$	$(\hat{k},\tilde{d})$	$(\hat{k}_{AM},\tilde{d})$
				$\sigma = 1$				
50	0.9	1.4150	0.1158	0.0746	**0.0603**	0.0612	0.1137	0.1478
	0.95	5.0337	0.2443	0.1291	0.0925	**0.0862**	0.1708	0.2469
	0.99	28.8521	1.2653	0.6240	0.3902	0.3388	**0.3028**	0.4972
	0.999	313.1557	13.1267	6.1288	3.5476	2.9779	**0.8574**	5.1651
100	0.9	0.6317	**0.0544**	0.0595	0.0697	0.0694	0.0771	0.0775
	0.95	2.0512	0.1103	0.0816	0.0780	**0.0749**	0.1179	0.1402
	0.99	6.2194	0.4740	0.2327	0.1363	**0.1315**	0.1753	0.2200
	0.999	64.2228	4.7773	2.1764	1.2076	1.0481	**0.5705**	31.7646
150	0.9	0.2552	0.0324	**0.0263**	0.0329	0.0336	0.0294	0.0294
	0.95	0.5564	0.0576	0.0351	**0.0295**	0.0320	0.0366	0.0372
	0.99	2.7344	0.2779	0.1381	**0.0839**	0.0895	0.1184	0.1386
	0.999	27.2799	2.8166	1.2591	0.6948	0.6151	**0.4353**	2.4974
200	0.9	0.0807	0.0195	**0.0126**	0.0197	0.0194	0.0232	0.0232
	0.95	0.1915	0.0372	**0.0196**	0.0255	0.0252	0.0267	0.0266
	0.99	0.9097	0.1686	0.0756	**0.0566**	0.0599	0.0594	0.0583
	0.999	9.2712	1.7184	0.6854	0.2489	0.2453	**0.2404**	0.2546
				$\sigma = 5$				
50	0.9	35.5259	2.8791	1.4423	1.1950	1.1254	**0.6321**	38.6642
	0.95	125.2065	5.9177	2.7826	2.0007	1.7792	**0.6932**	15.8366

T	λ	$\hat{\alpha}$	$\hat{\alpha}_M$	$\hat{\alpha}_M(k)$	$\hat{\alpha}_M(k,d)$		$\tilde{\alpha}_M(k,d)$	
					$(\hat{k},\hat{d})$	$(\hat{k}_{AM},\hat{d})$	$(\hat{k},\tilde{d})$	$(\hat{k}_{AM},\tilde{d})$
	0.99	722.1768	31.1187	14.3612	8.1765	6.7604	**1.5824**	109.5636
	0.999	7824.2789	330.6736	153.8738	88.3979	74.1219	**13.9455**	928.7646
100	0.9	15.6956	1.2975	0.7029	0.6272	0.5897	**0.4529**	1.6065
	0.95	51.6175	2.6411	1.3079	1.0088	0.9256	**0.5209**	44.7475
	0.99	155.2863	11.6694	5.2834	2.7509	2.3329	**0.8651**	68.6101
	0.999	1603.7042	115.8095	51.7146	28.0082	23.6957	**5.8923**	140.8472
150	0.9	6.2517	0.7908	0.4340	0.3322	**0.3204**	0.3327	0.5972
	0.95	13.8289	1.4309	0.6845	0.4530	0.4180	**0.3603**	0.7623
	0.99	68.8059	6.9658	3.0411	1.6228	1.3817	**0.6684**	7.5994
	0.999	676.4298	69.1994	29.6038	16.2876	13.7963	**3.6296**	98.3212
200	0.9	2.0266	0.5019	0.2494	**0.1591**	0.1600	0.1815	0.2115
	0.95	4.8099	0.9016	0.3915	0.2091	**0.2062**	0.2140	0.2482
	0.99	23.3909	4.4504	1.7514	0.5640	0.5114	**0.3435**	0.4342
	0.999	234.6549	43.6282	16.8014	4.7894	4.1585	**1.2534**	158.9998
				$\sigma = 10$				
50	0.9	142.1743	11.5622	5.6318	4.7364	4.4242	**1.4163**	616.6213
	0.95	503.5359	23.8605	11.1796	7.9914	7.0962	**1.7567**	3107.4833
	0.99	2896.1165	127.4229	59.2648	33.7123	28.1181	**5.2061**	284.6411
	0.999	31264.2819	1297.1003	599.3104	354.2013	296.8510	**54.2811**	1998.1696
100	0.9	63.4880	5.3784	2.7585	2.3965	2.2560	**1.0053**	25.5626
	0.95	205.0308	10.5446	5.1247	3.9242	3.6197	**1.3732**	171.7053
	0.99	619.1007	47.0466	20.9263	10.5351	8.8110	**2.3229**	203.1956
	0.999	6380.2122	464.2539	205.4810	113.6608	95.7406	**21.2887**	6035.4811
150	0.9	24.9874	3.1250	1.5735	1.2371	1.1642	**0.7086**	8.5602
	0.95	55.6497	5.8048	2.6362	1.7235	1.5544	**0.7568**	6.8451
	0.99	275.7269	28.2547	12.1183	6.4718	5.4712	**1.6477**	46.8435
	0.999	2736.5701	282.7309	119.5827	62.4642	52.9023	**12.6865**	1070.5062
200	0.9	8.2269	2.0299	0.9248	0.5728	0.5189	**0.4039**	0.9837
	0.95	18.9001	3.6432	1.5080	0.7515	0.6640	**0.4420**	0.8140
	0.99	92.0745	17.2901	6.6607	2.0897	1.8286	**0.6940**	5.3643
	0.999	928.5438	171.2112	65.5820	18.1525	15.6254	**3.4167**	128.9422

Table 4. Estimated MAE values of the estimators with one outlier.

T	λ	$\hat{\alpha}$	$\hat{\alpha}_M$	$\hat{\alpha}_M(k)$	$\hat{\alpha}_M(k,d)$		$\tilde{\alpha}_M(k,d)$	
					$(\hat{k},\hat{d})$	$(\hat{k}_{AM},\hat{d})$	$(\hat{k},\tilde{d})$	$(\hat{k}_{AM},\tilde{d})$
				$\sigma = 1$				
50	0.9	1.4167	0.7277	0.6142	**0.5665**	0.5667	0.7136	0.8435
	0.95	2.1854	0.9962	0.7571	0.7132	**0.6812**	1.0049	1.3721
	0.99	5.3811	2.1882	1.5347	1.3235	**1.2633**	1.4134	1.8378
	0.999	17.5246	6.9470	4.5391	4.1369	3.6908	**2.5283**	4.3001
100	0.9	0.8656	**0.5314**	0.5547	0.6608	0.6628	0.6680	0.6733
	0.95	1.4295	0.7292	**0.6623**	0.6749	0.6705	0.8701	0.9377
	0.99	3.0024	1.5065	1.0588	**0.8656**	0.8819	1.0155	1.1878
	0.999	9.6459	4.7249	3.1496	2.6603	2.4622	**2.2101**	3.2835
150	0.9	0.6367	0.4180	**0.4178**	0.4840	0.4895	0.4746	0.4782
	0.95	0.8729	0.5515	0.4458	**0.4160**	0.4365	0.4782	0.4907

T	λ	$\hat{\alpha}$	$\hat{\alpha}_M$	$\hat{\alpha}_M(k)$	$\hat{\alpha}_M(k,d)$		$\tilde{\alpha}_M(k,d)$	
					$(\hat{k},\hat{d})$	$(\hat{k}_{AM},\hat{d})$	$(\hat{k},\tilde{d})$	$(\hat{k}_{AM},\tilde{d})$
	0.99	2.0296	1.1933	0.8505	**0.6915**	0.7369	0.8202	0.8998
	0.999	6.1735	3.7896	2.4866	2.0227	1.9093	**1.8016**	2.2841
200	0.9	0.4669	0.3339	**0.2756**	0.3383	0.3352	0.3561	0.3589
	0.95	0.6294	0.4428	**0.3234**	0.3854	0.3831	0.3717	0.3744
	0.99	1.2808	0.9476	0.6170	0.5558	0.5654	**0.5433**	**0.5433**
	0.999	3.9532	3.0228	1.8077	**1.1316**	1.1759	1.1714	1.2233
				$\sigma = 5$				
50	0.9	7.1194	3.6020	2.5755	2.5451	2.4424	**2.1622**	4.9003
	0.95	11.0112	4.8897	3.3500	3.2495	3.0339	**2.2676**	5.5963
	0.99	27.0629	11.0211	7.2168	6.3679	5.6467	**3.5687**	9.2794
	0.999	87.7097	35.0589	22.8826	21.0742	18.7748	**8.7956**	25.2460
100	0.9	4.2970	2.6487	2.0288	1.9532	1.9083	**1.8548**	2.6422
	0.95	7.0652	3.5690	2.5533	2.4682	2.3645	**1.8974**	2.8339
	0.99	14.9449	7.4607	4.8784	4.0438	3.6495	**2.6860**	5.2987
	0.999	48.6428	24.2621	15.7682	13.3779	12.0822	**6.2509**	13.3811
150	0.9	3.1530	2.0868	1.5489	1.4231	**1.3795**	1.4426	1.7308
	0.95	4.3856	2.7576	1.8894	1.6224	**1.5410**	1.5419	1.9591
	0.99	10.0775	5.9916	3.8537	3.1277	2.8741	**2.3139**	3.5012
	0.999	30.7991	18.9898	11.9963	9.8849	8.9704	**4.8557**	10.3846
200	0.9	2.3452	1.6807	1.1714	**0.9482**	0.9604	1.0353	1.1087
	0.95	3.1329	2.2271	1.4385	**1.0684**	1.0898	1.1196	1.2130
	0.99	6.4168	4.7851	2.8105	1.7444	1.6941	**1.5000**	1.6768
	0.999	19.4569	14.9652	8.6618	5.2945	4.7898	**2.9200**	11.9621
				$\sigma = 10$				
50	0.9	14.1004	7.1886	4.9337	5.0056	4.7853	**3.0372**	8.6607
	0.95	21.9032	9.7564	6.6225	6.5145	6.0744	**3.4649**	10.8446
	0.99	53.8622	21.7247	14.0436	12.5331	10.9810	**5.7296**	19.9349
	0.999	175.9525	70.0277	45.6664	42.4009	37.7047	**17.0412**	50.1240
100	0.9	8.6352	5.2577	3.8184	3.7769	3.6691	**2.6516**	5.8079
	0.95	14.1387	7.2163	5.0161	4.8450	4.6327	**2.8298**	5.8634
	0.99	29.7880	14.8870	9.6505	7.9593	7.1305	**4.1132**	11.2553
	0.999	96.2950	47.3892	30.8604	26.4389	23.8282	**11.2790**	27.7205
150	0.9	6.3265	4.1596	2.9206	2.7464	2.6396	**2.1959**	3.5919
	0.95	8.6894	5.4457	3.6065	3.2120	3.0090	**2.2988**	3.7826
	0.99	20.3979	12.1181	7.7347	6.3297	5.7508	**3.4881**	6.6679
	0.999	62.0041	38.0257	24.1861	20.0762	18.1761	**8.4698**	23.4670
200	0.9	4.6632	3.3858	2.2214	1.8279	1.7315	**1.6207**	1.9183
	0.95	6.3316	4.4239	2.7425	2.0505	1.9238	**1.7038**	2.1230
	0.99	12.6652	9.5406	5.5548	3.4525	3.1668	**2.2457**	3.5088
	0.999	39.6518	30.5550	17.6731	10.5541	9.4639	**4.6223**	8.7188

Table 5. Estimated MAE values of the estimators with two outliers.

T	λ	$\hat{\alpha}$	$\hat{\alpha}_M$	$\hat{\alpha}_M(k)$	$\hat{\alpha}_M(k,d)$		$\tilde{\alpha}_M(k,d)$	
					$(\hat{k},\hat{d})$	$(\hat{k}_{AM},\hat{d})$	$(\hat{k},\tilde{d})$	$(\hat{k}_{AM},\tilde{d})$
				$\sigma = 1$				
50	0.9	3.7728	0.7665	0.6882	**0.6424**	0.6524	0.8129	0.9502

T	λ	$\hat{\alpha}$	$\hat{\alpha}_M$	$\hat{\alpha}_M(k)$	$\hat{\alpha}_M(k,d)$		$\tilde{\alpha}_M(k,d)$	
					$(\hat{k},\hat{d})$	$(\hat{k}_{AM},\hat{d})$	$(\hat{k},\tilde{d})$	$(\hat{k}_{AM},\tilde{d})$
	0.95	5.8218	1.0576	0.8580	0.8060	**0.7813**	1.1032	1.3986
	0.99	9.9843	2.2507	1.6562	1.4516	**1.3878**	1.5155	1.8764
	0.999	34.1555	7.2548	4.9994	4.4629	4.0326	**2.6185**	4.3227
100	0.9	2.3875	0.6923	0.7937	**0.6832**	0.6842	0.6863	0.6881
	0.95	3.7501	0.7707	0.7018	0.7056	**0.6960**	0.8394	0.8967
	0.99	4.0944	1.5217	1.0872	**0.8944**	0.9070	1.0537	1.2347
	0.999	13.5874	4.8571	3.2814	2.7555	2.5611	**2.2356**	3.3038
150	0.9	1.2297	0.4243	**0.4146**	0.4887	0.4937	0.4527	0.4574
	0.95	1.4997	0.5613	0.4592	**0.4386**	0.4597	0.4862	0.5019
	0.99	2.7393	1.1958	0.8578	**0.7076**	0.7529	0.8510	0.9306
	0.999	8.5218	3.8168	2.5065	2.0343	1.9168	**1.8173**	2.3485
200	0.9	0.5592	0.3317	**0.2745**	0.3457	0.3425	0.3439	0.3473
	0.95	0.7381	0.4470	**0.3287**	0.3979	0.3954	0.3672	0.3706
	0.99	1.4077	0.9554	0.6307	0.5786	0.5904	0.5751	**0.5736**
	0.999	4.4220	3.0422	1.8511	**1.1858**	1.2233	1.2214	1.2790
				$\sigma = 5$				
50	0.9	18.9029	3.7798	2.9168	2.8489	2.7698	**2.5522**	5.0532
	0.95	29.1703	5.1547	3.8127	3.6794	3.4789	**2.5707**	4.9631
	0.99	50.0773	11.3464	7.8682	6.9241	6.2489	**3.7367**	8.9859
	0.999	170.9169	36.7916	25.3696	22.9242	20.6744	**9.7232**	19.5715
100	0.9	11.9446	2.7611	2.0986	2.0598	2.0036	**1.8185**	2.4082
	0.95	18.8311	3.7981	2.7541	2.6718	2.5600	**1.9746**	2.8433
	0.99	20.4289	7.5178	4.9440	4.0664	3.6844	**2.6691**	4.9493
	0.999	67.9247	24.0942	15.8255	13.5192	12.2569	**6.3338**	12.9527
150	0.9	6.1553	2.1167	1.6150	1.4766	**1.4438**	1.5863	2.0000
	0.95	7.4407	2.7961	1.9567	1.7025	**1.6349**	1.6862	2.1831
	0.99	13.7159	6.0107	3.8901	3.2022	2.9438	**2.3473**	4.0112
	0.999	42.8329	19.0426	12.0556	9.9888	9.0890	**4.8954**	11.3806
200	0.9	2.8242	1.6702	1.1636	**0.9511**	0.9634	1.0488	1.1353
	0.95	3.6319	2.1836	1.4174	**1.0788**	1.1028	1.1487	1.2543
	0.99	7.1317	4.8838	2.9012	1.8034	1.7488	**1.5281**	1.7009
	0.999	22.2260	14.9115	8.6089	5.2533	4.7503	**2.9161**	4.7570
				$\sigma = 10$				
50	0.9	37.6656	7.5179	5.5858	5.5583	5.3764	**3.6462**	8.5465
	0.95	58.3141	10.4001	7.6188	7.3849	6.9713	**4.1191**	8.9657
	0.99	99.8673	22.5376	15.3608	13.4877	12.0490	**6.0968**	14.0644
	0.999	341.8865	72.2373	49.8005	45.5458	41.0536	**18.5460**	42.7439
100	0.9	23.9925	5.5008	4.0217	4.0174	3.8956	**2.7049**	5.4157
	0.95	37.4800	7.5297	5.3808	5.2765	5.0543	**3.0971**	5.3023
	0.99	40.8857	15.1475	9.9196	8.2199	7.3839	**4.0996**	8.3745
	0.999	135.6486	48.0802	31.3176	26.6111	24.0404	**10.8642**	28.5779
150	0.9	12.3505	4.2408	3.0593	2.8594	2.7641	**2.4089**	4.1598
	0.95	14.9695	5.5390	3.7250	3.3077	3.1244	**2.4918**	5.1327
	0.99	27.5046	12.1226	7.7644	6.3701	5.7882	**3.5603**	7.0979
	0.999	85.5882	38.4197	24.3858	20.1707	18.2507	**8.4911**	19.3821
200	0.9	5.6226	3.3210	2.2143	1.8394	1.7467	**1.6801**	2.0922

T	λ	$\hat{\alpha}$	$\hat{\alpha}_M$	$\hat{\alpha}_M(k)$	$\hat{\alpha}_M(k,d)$		$\tilde{\alpha}_M(k,d)$	
					$(\hat{k},\hat{d})$	$(\hat{k}_{AM},\hat{d})$	$(\hat{k},\tilde{d})$	$(\hat{k}_{AM},\tilde{d})$
	0.95	7.3927	4.4755	2.8024	2.1164	1.9874	**1.7851**	2.2220
	0.99	14.1173	9.5934	5.5944	3.4499	3.1598	**2.2534**	3.4674
	0.999	44.5182	30.4312	17.5715	10.5797	9.4836	**4.5808**	9.3946

Table 6. Estimated MAE values of the estimators with four outliers.

T	λ	$\hat{\alpha}$	$\hat{\alpha}_M$	$\hat{\alpha}_M(k)$	$\hat{\alpha}_M(k,d)$		$\tilde{\alpha}_M(k,d)$	
					$(\hat{k},\hat{d})$	$(\hat{k}_{AM},\hat{d})$	$(\hat{k},\tilde{d})$	$(\hat{k}_{AM},\tilde{d})$
				$\sigma = 1$				
50	0.9	2.8849	0.8355	0.6839	**0.6179**	0.6269	0.8280	0.9728
	0.95	5.3570	1.1944	0.8901	0.7922	**0.7664**	1.0850	1.3439
	0.99	11.7120	2.7032	1.9152	1.6027	**1.5164**	1.5726	1.9576
	0.999	38.4250	8.6679	5.7617	4.7511	4.2740	**2.4710**	3.7576
100	0.9	1.9692	0.7518	0.7127	**0.7061**	0.7076	0.7350	0.7444
	0.95	3.5035	0.8118	0.7263	0.7203	**0.7091**	0.8617	0.9667
	0.99	5.7941	1.6607	1.1549	**0.9151**	0.9222	1.0817	1.2509
	0.999	18.7217	5.2931	3.4897	2.7790	2.5793	**2.1808**	3.2318
150	0.9	1.2015	0.4401	**0.4091**	0.4858	0.4902	0.4342	0.4364
	0.95	1.7833	0.5865	0.4674	**0.4466**	0.4672	0.4766	0.4877
	0.99	3.9258	1.2926	0.9230	**0.7437**	0.7863	0.8889	0.9728
	0.999	12.5470	4.1234	2.6727	2.0571	1.9458	**1.8326**	2.3081
200	0.9	0.6999	0.3440	**0.2792**	0.3678	0.3647	0.3325	0.3342
	0.95	1.1004	0.4641	**0.3388**	0.4258	0.4232	0.3681	0.3697
	0.99	2.4867	1.0260	0.6727	0.6142	0.6277	0.5955	**0.5924**
	0.999	7.9036	3.2212	1.9309	**1.2239**	1.2558	1.2713	1.3170
				$\sigma = 5$				
50	0.9	14.4066	4.2051	2.9855	2.8240	2.7398	**2.2392**	4.2721
	0.95	26.8271	5.9655	4.0937	3.6698	3.4583	**2.2637**	4.6671
	0.99	58.5491	13.6279	9.0261	7.2830	6.5642	**3.3818**	8.4583
	0.999	192.4635	43.8998	29.3025	24.4622	22.1355	**8.4176**	23.1100
100	0.9	9.7480	2.8691	2.1394	2.0582	1.9974	**1.8506**	2.6806
	0.95	17.4884	3.9531	2.7508	2.5657	2.4534	**1.9133**	3.1829
	0.99	28.8609	8.3734	5.3657	4.0830	3.7035	**2.5353**	4.4162
	0.999	94.0482	26.8008	17.3105	13.6715	12.3984	**5.7425**	12.7439
150	0.9	6.0212	2.1871	1.6407	1.4714	**1.4438**	1.5775	1.9034
	0.95	8.9341	2.9188	2.0011	1.6831	**1.6224**	1.6626	2.1583
	0.99	19.6124	6.4829	4.1520	3.2521	2.9903	**2.3289**	3.7911
	0.999	62.2968	20.3770	12.8722	10.1207	9.2014	**4.6937**	10.0561
200	0.9	3.4778	1.7411	1.2276	**1.0000**	1.0178	1.1200	1.2069
	0.95	5.4718	2.3383	1.5086	**1.1353**	1.1535	1.2134	1.3226
	0.99	12.3729	5.1115	3.0183	1.8343	1.7646	**1.5739**	1.7274
	0.999	39.2369	15.9738	9.2599	5.3533	4.8418	**2.8961**	4.7237
				$\sigma = 10$				
50	0.9	28.9174	8.4148	5.8245	5.6283	5.4342	**3.2010**	9.5642
	0.95	53.5802	11.9441	8.0777	7.3056	6.8627	**3.3607**	9.0279
	0.99	117.3331	27.1932	18.0522	14.7862	13.2429	**5.4324**	13.5444
	0.999	385.0803	87.5682	58.0905	48.6870	44.1454	**15.8831**	42.5833

T	λ	$\hat{\alpha}$	$\hat{\alpha}_M$	$\hat{\alpha}_M(k)$	$\hat{\alpha}_M(k,d)$		$\tilde{\alpha}_M(k,d)$	
					$(\hat{k},\hat{d})$	$(\hat{k}_{AM},\hat{d})$	$(\hat{k},\tilde{d})$	$(\hat{k}_{AM},\tilde{d})$
100	0.9	19.4439	5.7274	4.0665	3.9577	3.8344	**2.7128**	5.5862
	0.95	34.9280	7.9512	5.4652	5.0818	4.8642	**2.8837**	5.8239
	0.99	57.5862	16.4677	10.5180	8.1612	7.3396	**3.8568**	9.2711
	0.999	188.1785	52.7794	33.7584	26.6391	24.1223	**9.8310**	29.7708
150	0.9	12.1259	4.4308	3.1451	2.8712	2.7860	**2.3452**	4.0127
	0.95	17.9038	5.9060	3.9398	3.3888	3.2116	**2.4603**	4.1613
	0.99	39.4194	13.1013	8.3406	6.4423	5.8863	**3.4696**	7.0454
	0.999	125.3975	41.2827	26.2204	20.3907	18.6225	**8.1911**	19.7653
200	0.9	6.9468	3.4603	2.3035	1.8878	1.8035	**1.7389**	2.1104
	0.95	11.0043	4.7497	2.9347	2.1616	2.0335	**1.8263**	2.2174
	0.99	24.7752	10.2813	5.9534	3.4911	3.1879	**2.2581**	3.7389
	0.999	78.9935	32.0443	18.4672	10.5992	9.4924	**4.5843**	9.0068

5.2. Results and Discussion

Tables 1-3 display the estimated MSE values while Tables 4-6 show the estimated MAE values of the estimators under study for various simulation settings. For all the tables, the bold values indicate the smallest MSE/MAE values in each row. The following conclusions can be drawn from the results obtained from the Monte Carlo simulations.

- MSE of our proposed ALTME remains less than that of the AE, AME and ARME in almost all cases except for the cases when $\sigma = 1$ along with λ = 0.9 and T = 100, 150 and 200 where ARME performs better. However, ALTME always gives superb performance in the sense of smaller MSE when $\sigma = 5$ and 10 for all simulation settings.
- The increase in the number of outlier cases severely affects the MSE of all the estimators. It is observed that MSE of all the estimators increases by increasing the number of outlier cases. However, our proposed ALTME remains less sensitive to increase in the number of outlier cases.
- The increase in the collinearity level results in a very adverse impact on MSE of all the estimators. MSE of all the estimators show significant increase due to increase in collinearity level.
- MSE of all the estimators decreases with the increase in sample size T which is in the line of what the literature suggests.
- We evaluated the performance of estimators for three cases of error standard deviation i.e. $\sigma = 1$, 5 and 10. It is noted that increase in the value of σ causes the substantial increase in MSE of all the estimators. However, the performance of our proposed ALTME improves when the value of error standard deviation increases. When $\sigma = 5$ and 10, we note that our proposed estimator $\tilde{\alpha}_M(k,d)$ with the combination of biasing parameter estimators $(\hat{k},\tilde{d})$ performs better than the AE, AME and ARME in almost all simulation settings.
- The ALTME performs better for higher values of σ because of the way it balances **bias and variance** through its biasing parameters $(\hat{k},\tilde{d})$. When σ is large, traditional estimators such as AE, AME, and ARME experience a sharp increase in variance, which leads to much larger MSE. In contrast, ALTME incorporates additional shrinkage (through its biasing structure), which controls the variance inflation more effectively.

The findings indicate that across all cases with a standard deviation value of 1, no individual estimator demonstrates a statistically significant advantage in terms of generating smaller MSE. This observation holds true for various levels of collinearity, sample sizes, and different outlier cases.

6. Applications

In this section, two real-life datasets are used to illustrate the performance of the AE, the AME, the ARME, and our proposed ALTME. The use of real-life datasets allows us to assess the performance of these estimators under practical conditions, which often involve complexities such as multicollinearity and outliers. The simulation results presented earlier have already demonstrated the effectiveness and robustness of our proposed estimators. To further validate their performance in real-life scenarios, we use the MSE criterion to compare the performance and reliability of estimators under study.

Example 1: The Almon dataset

The Almon dataset is used to illustrate the performance of estimators under practical conditions. This data set consists of quarterly records for the period of 1959 to 1967. In the Almon dataset *capital expenditures* are considered as the dependent variable and *appropriations* as the independent variable. Following Özbay and Kaçıranlar (2017) and Majid et al. (2024), we use the lag length p = 8 and the degree of polynomial r = 2. The matrix Z in Eq. (6) is obtained as $Z = XR$, where R is defined in Eq. (5). The condition number of $Z'Z$ for this dataset is 4033 indicating the presence of severe multicollinearity. Majid and Aslam (2023) and Majid et al. (2024) showed that there are no outliers in y-direction in this dataset. The model in Eq. (6) is estimated by using estimators under study. The MSE of the AE, the AME, the ARME and proposed estimator the ALTME are calculated using Eq. (10)-(15), respectively.

In Table 7, the estimated coefficients and the MSEs of estimators are presented. It is observed that our proposed estimator $\tilde{\alpha}_{ALTME}(\hat{k}_{AM}, \tilde{d})$ gives the smaller MSE as compared to the AE, the AME, and the ARME, which is clearly indicating that our proposed estimator is more robust and yields more accurate and reliable results in the presence of severe multicollinearity. The superb performance of $\tilde{\alpha}_{ALTME}(\hat{k}_{AM}, \tilde{d})$ highlights its effectiveness in handling multicollinearity issues in the DLM.

To assess the performance of estimators in the presence of outliers, some outliers are introduced in the dataset. Following the work of Silvapulle (1991) and Majid et al. (2024) the outliers are introduced by altering the y-values for observations 2, 14 and 32 such as $y_2^* = y_2 + 10\hat{\sigma}$, $y_{14}^* = y_{14} + 15\hat{\sigma}$ and $y_{27}^* = y_{27} + 5\hat{\sigma}$, where asterisk indicates the new value and $\hat{\sigma}$ is the standard deviation of the residuals obtained by the AE corresponding to the original dataset.

In Table 8, estimated coefficients and the MSE values of the estimators under study, are presented in the presence of outliers. It is observed that MSE of the AE has increased dramatically after introducing outliers. The MSE of AE is much larger than the MSE of AME, which clearly indicates the robustness of the AME over the AE. However, it is worth noting that our proposed estimator $\tilde{\boldsymbol{\alpha}}_{\boldsymbol{M}}(\hat{\boldsymbol{k}}_{\boldsymbol{AM}}, \tilde{\boldsymbol{d}})$ outperforms not only the AME but all other estimators by providing minimum value of MSE in the presence of joint issue of multicollinearity and outlier.

Table 7. MSE Comparison of different Estimators for Almon dataset with no outlier

Parameter	$\hat{\alpha}$	$\hat{\alpha}_M$	$\hat{\alpha}_M(k)$	$\hat{\alpha}_M(k,d)$		$\tilde{\alpha}_M(k,d)$	
				$(\hat{k}, \hat{d})$	$(\hat{k}_{AM}, \hat{d})$	$(\hat{k}, \tilde{d})$	$(\hat{k}_{AM}, \tilde{d})$
γ_1	-3.9850	-3.9770	-3.9770	-3.9700	-4.0100	-3.9400	-3.9570
γ_2	7.6800	7.9650	7.8840	7.4560	8.6440	5.4700	6.3080
γ_3	-8.7820	-9.4170	-4.9520	-7.5430	-0.6490	-0.2310	-0.3940
MSE	16.5600	15.3640	24.3140	13.6710	117.0760	13.6600	**13.2150**

Table 8. MSE Comparison of different Estimators for Almon dataset with outliers

Parameter	$\hat{\alpha}$	$\hat{\alpha}_M$	$\hat{\alpha}_M(k)$	$\hat{\alpha}_M(k,d)$		$\tilde{\alpha}_M(k,d)$	
				$(\hat{k}, \hat{d})$	$(\hat{k}_{AM}, \hat{d})$	$(\hat{k}, \tilde{d})$	$(\hat{k}_{AM}, \tilde{d})$
γ_1	-3.8253	-3.8342	-3.8331	-3.8235	-3.8641	-3.7978	-3.8110
γ_2	5.8024	7.2543	7.1568	6.6063	7.8171	4.9817	5.5707
γ_3	-9.3659	-8.9676	-4.1021	-6.4734	-0.6104	-0.2202	-0.3282
MSE	165.6962	18.62247	15.9210	16.4122	114.2675	16.5089	**15.6528**

Example 2: The annual data of GDP and total investment (1973-2015) for Pakistan

The second dataset was pertaining to Pakistan's economy. The same dataset is already used by Majid et al. (2024). The dataset consists of annual gross domestic product (GDP) and total investments in the country for the period 1973 to 2015. The data on GDP and total investment is presented in million rupees at current prices. The GDP is taken as the dependent variable (y) and the investments as the independent variable (x). It is the intuition that the GDP in a particular year t may be influenced not only by the current year's investment but also by the investments made in previous years i.e.

$$y_t = \beta_0 x_t + \beta_1 x_{t-1} + \ldots + \beta_p x_{t-p} + u_t, \qquad t = p+1, \ldots, T$$

The interest is in estimating the above model. For this purpose, we use all the estimators under study. Following Majid et al. (2024), the lag length $p = 8$ and the degree of polynomial $r = 2$ is taken. The matrix Z in Model (6) is obtained by $Z =$

XR, where R is defined in Eq. (5). The presence of multicollinearity is examined by calculating the condition numbers of matrix $Z'Z$. The condition number of matrix $Z'Z$ is 11381, suggesting the presence of severe multicollinearity. Majid et al. (2024) showed the presence of outliers in dataset.

Table 9. MSE Comparison under different Estimators for GDP dataset

Parameter	$\hat{\alpha}$	$\hat{\alpha}_M$	$\hat{\alpha}_M(k)$	$\hat{\alpha}_M(k,d)$		$\tilde{\alpha}_M(k,d)$	
				$(\hat{k},\hat{d})$	$(\hat{k}_{AM},\hat{d})$	$(\hat{k},\tilde{d})$	$(\hat{k}_{AM},\tilde{d})$
γ_1	-3.3660	-3.3620	-3.3620	-3.3610	-3.3290	-3.4730	-3.3590
γ_2	3.1120	3.0840	3.0770	2.9750	0.7670	3.2790	2.3910
γ_3	-6.3950	-4.4280	-4.0450	-4.2210	-0.0330	-0.1770	-0.3240
MSE	3.5600	1.0930	1.0630	1.0490	**1.0390**	298.4560	**1.0390**

In Table 9, estimated coefficient and the MSE values of the estimators under study are presented. The dataset is contaminated with severe multicollinearity and the presence of outliers. The results indicate that both of our proposed estimators significantly outperform the AE, AME, and ARME by providing the smallest MSE. So, it can be concluded that $\hat{\alpha}_M(\hat{k}_{AM},\hat{d})$ and $\tilde{\alpha}_M(\hat{k}_{AM},\tilde{d})$ are more robust and better to handle the complexities of multicollinearity and outliers more effectively than the other estimators.

7. Conclusion

This study addresses the robust estimation of the DLM using the Almon technique under the joint challenges of multicollinearity and outliers in the y-direction. We proposed the ALTME for the parameter vector of lag coefficients. Both theoretical analysis and extensive evidence from Monte Carlo simulations and real-world datasets confirm the superiority of ALTME. In particular, ALTME consistently achieves substantially lower MSE than conventional estimators such as AE, AME, and ARME, especially under conditions of high error variance, moderate to severe multicollinearity, and data contamination by outliers. Given these promising results, future research could extend ALTME to broader econometric frameworks, such as dynamic panel models, time-varying coefficient models, or nonlinear specifications, to further assess its robustness and applicability in more complex empirical settings.

REFERENCES

Almon, S. (1965). The distributed lag between capital appropriations and expenditures. Econometrica: Journal of the Econometric Society, 178-196.

Ertaş, H. (2017). A modified ridge m-estimator for linear regression model with multicollinearity and outliers. Communications in Statistics - Simulation and Computation, 47(4), 1240-1250. https://doi.org/10.1080/03610918.2017.1310231

Ertaş, H., Kaçıranlar, S., & Güler, H. (2017). Robust Liu-type estimator for regression based on M-estimator. Communications in Statistics-Simulation and Computation, 46(5), 3907-3932.

Ertaş, H., Toker, S., & Kaçıranlar, S. (2015). Robust two parameter ridge M-estimator for linear regression. Jour-nal of Applied Statistics, 42(7), 1490-1502. https://doi.org/10.1080/02664763.2014.1000577

Fair, R. C., & Jaffee, M. (1971). A note on the estimation of polynomial distributed lags. Princeton University.

Frost, P. A. (1975). Some properties of the Almon lag technique when one searches for degree of polynomial and lag. Journal of the American Statistical Association, 70(351a), 606-612.

Güler, H., Gültay, B., & Kaçiranlar, S. (2017). Comparisons of the alternative biased estimators for the distributed lag models. Communications in Statistics-Simulation and Computation, 46(4), 3306-3318.

Gültay, B., & Kaçiranlar, S. (2015). Mean square error comparisons of the alternative estimators for the distribut-ed lag models. Hacettepe Journal of Mathematics and Statistics, 44(5), 1215-1233.

Hoerl, A. E., & Kennard, R. W. (1970). Ridge regression: Biased estimation for nonorthogonal problems. Tech-nometrics, 12(1), 55-67. https://doi.org/10.1080/00401706.1970.10488634

Huber, P. (1981). Robust statistics. John Wiley and Sons.

Judge, G. G., Griffiths, W. E., Hill, R. C., & Lutkepohl, H. (1980). Lee, Tsoung-Chao. The Theory and Practice of Econometrics. New York: John Wiley and Sons.

Kibria, B. M. G. (2003). Performance of some new ridge regression estimators. Communications in Statistics-Simulation and Computation, 32(2), 419-435.

Kibria, B. M. G. (2022). More than hundred (100) estimators for estimating the shrinkage parameter in a linear and generalized linear ridge regression models. Journal of Econometrics and Statistics, 2(2), 233-252.

Liu, K. (2003). Using Liu-type estimator to combat collinearity. Communications in Statistics-Theory and Meth-ods, 32(5), 1009-1020.

Lukman, A, and Kibria, B. M. G. (2021). Almon-KL estimator for the Distributed Lag Model. Arab Journal of Basic and Applied Sciences, 28(1), 406-412.
Maddala, G. S. 1977. Econometrics. New York: McGraw-Hill.
Majid, A., & Aslam, M. (2023). The Almon M-estimator for the distributed lag model in the presence of outli-ers. Communications in Statistics-Simulation and Computation, 52(7), 3273-3285.
Majid, A., Aslam, M., Ahmad, S., Altaf, S., & Afzal, S. (2024). Robust estimation of the distributed lag model with multicollinearity and outliers. Communications in Statistics-Simulation and Computation, 53(8), 3933-3947.
Månsson, K., Shukur, G., & Kibria, B. M. G. (2010). A simulation study of some ridge regression estimators un-der different distributional assumptions. Communications in Statistics - Simulation and Computation, 39(8), 1639-1670. https://doi.org/10.1080/03610918.2010.508862
Özbay, N. (2019). Two-parameter ridge estimation for the coefficients of Almon distributed lag model. Iranian Journal of Science and Technology, Transactions A: Science, 43, 1819-1828.
Özbay, N., & Kaçıranlar, S. (2016). The Almon two parameter estimator for the distributed lag models. Journal of Statistical Computation and Simulation, 87(4), 834-843. https://doi.org/10.1080/00949655.2016.1229317
Özbay, N., & Kaçıranlar, S. (2017). The Almon two parameter estimator for the distributed lag models. Journal of Statistical Computation and Simulation, 87(4), 834-843.
Qasim, M., Amin, M., & Omer, T. (2020). Performance of some new Liu parameters for the linear regression model. Communications in Statistics-Theory and Methods, 49(17), 4178-4196.
Silvapulle, M. J. (1991). Robust ridge regression based on an M-estimator. Australian Journal of Statistics, 33(3), 319-333.

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