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Estimation for a Regression Model based on Log Generalized Inverse Generalized Weibull distribution under Type I Censoring

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ABSTRACT

The Log Generalized Inverse Generalized Weibull (LGIGW) distribution, obtained by taking the logarithm of a Generalized Inverse Generalized Weibull (GIGW) random variable, is introduced and studied in this work. The proposed distribution is particularly useful for modeling lifetime, financial, and reliability data due to its flexibility in capturing various data behaviours. It encompasses several well-known classical distributions as special cases. In addition, a corresponding regression model based on the LGIGW distribution is developed. This model generalizes a number of existing regression models available in the literature, which arise as special cases under specific parameter settings. Parameter estimation is carried out using the method of maximum likelihood, and the performance of the estimators is assessed through Monte Carlo simulation studies.The applicability of the proposed model is illustrated using two real-life data sets. Model comparison is conducted using standard information criteria, including the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and the corrected Akaike Information Criterion (AICc). Furthermore, sensitivity analysis and residual diagnostic techniques are employed to identify influential observations.

Keywords: Regression, Censoring, Maximum Likelihood Estimation, Sensitivity Analysis, Residual Techniques

1 Introduction

Modeling lifetime data under censoring is a central problem in survival analysis, reliability theory, and biomedical research. Classical parametric models such as the Weibull distribution remain popular due to their flexibility and tractability. However, standard Weibull models are often inadequate for describing complex hazard rate shapes observed in practice, particularly in the presence of non-monotonic hazard functions. To address such limitations, several generalizations of the Weibull distribution have been proposed. The Inverse Weibull (IW) distribution introduced by Keller and Kamath (1982) has been widely used in reliability and survival studies. Subsequent works including Singh et al. (2013); Sultan et al. (2014); Nassar and Abo-Kasem (2017) investigated classical and Bayesian estimation under various censoring schemes. Krishna et al. (2019) further explored stress–strength reliability inference for the IW model. The logarithmic transformation of lifetime distributions has led to an important class of log-location-scale regression models. Kumar and Nair (2018) studied the Log Inverse Weibull (LIW) distribution whose applicability is limited in cases where the hazard function is non-decreasing. A major advancement was made by De Gusmao et al. (2011), who introduced the Generalized Inverse Weibull (GIW) and Log Generalized Inverse Weibull (LGIW) distributions, providing greater flexibility in modeling hazard behavior. More recently, Jain et al. (2014) proposed the Generalized Inverse Generalized Weibull (GIGW) distribution, which further extends the flexibility of inverse Weibull-type models. Motivated by this development, we introduce the Log Generalized Inverse Generalized Weibull (LGIGW) distribution, obtained by logarithmic transformation of a GIGW random variable. The proposed distribution accommodates increasing hazard rates and contains LGIW and LIW distributions as special cases. Other references in the area include Alsaggaf et al. (2024), Frolov et al. (2024), Aljeddani and Mohammed (2023), Temraz et al. (2022), El-Morshedy et al. (2022), Alotaibi et al. (2022).

There has been substantial progress in log-type regression modeling. Notable contributions include Log Birnbaum–Saunders regression model (Tabassum and Altaf (2025)) , Log-weighted exponential regression model (Altun (2021)) , Log-beta generalized half-normal regression model (Pescim et al. (2013)), Log-Kumaraswamy Generalized Gamma (Pascoa et al. (2013)), Log Generalized Modified Weibull (Ortega et al. (2011)), Log-Birnbaum-Saunders (Qu and Xie (2011)), Log-Exponentiated Weibull regression model (Hashimoto et al. (2010)).

Statistical inference in high-dimensional binary GLMs was addressed by Cai et al. (2023) and the SIHR package by Rakshit et al. (2021). The ordered beta regression model was proposed by Kubinec (2023). Penalized multinomial logistic regression with DP weights was developed in Fuetterer et al. (2025). A recent roadmap for logistic regression modeling was given in Bui and Ding (2025). Despite these advances, existing models may lack sufficient flexibility for modeling lifetime data exhibiting strong skewness. The LGIGW regression model proposed in this paper provides a unified and highly flexible framework encompassing several existing models as subcases. Specifically, the LGIW and LIW regression models arise as special cases when $\alpha = 1$ and $(\alpha, \gamma) = (1, 1)$, respectively. This hierarchical structure places the LGIGW model as a natural generalization of existing inverse Weibull-based regression frameworks.

The main contributions of this paper are Introduction of the LGIGW distribution and its properties, Development of the LGIGW regression model under Type I censoring, Derivation of likelihood equations and observed information matrix, Asymptotic inference including confidence interval construction, Simulation study for estimation of parameters. Empirical comparison with competing submodels, Sensitivity and residual diagnostics for detection of influential or extreme observations.For detection of influential observations, a lot of work has been done. Few recent references in this context are Khaleeq et al. (2021), Khan et al. (2025), Soale (2025).

The paper is organized as follows. In Section 2, the Log Generalized Inverse Generalized Weibull (LGIGW) distribution has been introduced. The corresponding regresssion model has been discussed in Section 3. Nonlinear equations for finding maximum likelihood estimates and the elements of information matrix have also been derived in this section. In Section 4, simulation study results are included. Section 5 consists of application to two real life data sets for which the proposed regression model has been compared with Log Generalized Inverse Weibull (LGIW) and Log Inverse Weibull (LIW) regression models using AIC, BIC and AICc. Section 6 discusses some sensitivity techniques viz global and local influence techniques under various perturbation schemes for identifying influential observations. To check the outliers, residual analysis is explained in Section 7. Conclusions are reported in Section 8.

2 Log Generalized Inverse Generalized Weibull Distribution

Let X be a random variable following Generalized Inverse Generalized Weibull (GIGW) distribution proposed by Jain et al. (2014). For $X \sim \text{GIGW}(\alpha, \gamma, \lambda, \beta)$, the density function of GIGW is given by

$$f(x) = \alpha\lambda^{\beta}\beta\gamma x^{-(\beta+1)}e^{-\gamma(\lambda/x)^{\beta}}(1 - e^{-\gamma(\lambda/x)^{\beta}})^{(\alpha-1)}; x > 0. \tag{2.1}$$

Then $Y = \log(X)$ follows Log Generalized Inverse Generalized Weibull (LGIGW) distribution. The probability density function (pdf) of Y is derived as

$$g(y) = \alpha\gamma\beta\lambda^{\beta}e^{-(\beta+1)y}e^{-\gamma\lambda^{\beta}e^{-y\beta}}\left\{1 - e^{-\gamma\lambda^{\beta}e^{-y\beta}}\right\}^{\alpha-1}e^{y}$$

$$= \alpha\gamma\beta\{e^{\beta\log\lambda - y\beta - \gamma e^{(\beta\log\lambda - y\beta)}}\}\{1 - e^{-\gamma e^{(\beta\log\lambda - y\beta)}}\}^{\alpha-1}, -\infty < y < \infty \tag{2.2}$$

Putting $\sigma = 1/\beta$ and $\mu = \log\lambda$ in (2.2), we write

$$g(y) = \frac{\alpha\gamma}{\sigma}e^{\left\{-(\frac{y-\mu}{\sigma}) - \gamma e^{\left[-(\frac{y-\mu}{\sigma})\right]}\right\}}\left\{1 - e^{-\gamma e^{\left[-(\frac{y-\mu}{\sigma})\right]}}\right\}^{\alpha-1}, -\infty < y < \infty \tag{2.3}$$

where $\alpha, \gamma, \sigma > 0$ and $-\infty < \mu < \infty$.

Accordingly, the LGIGW distribution is parameterized by $(\alpha, \gamma, \mu, \sigma)$, with parameter space given by

$$\alpha > 0, \quad \gamma > 0, \quad \sigma > 0, \quad \mu \in \mathbb{R}.$$

where $\sigma = 1/\beta$ and $\mu = \log \lambda$.
The corresponding survival function is

$$S(y) = \int_y^\infty \frac{\alpha\gamma}{\sigma} \exp\left\{-(\frac{m-\mu}{\sigma}) - \gamma \exp\left[-(\frac{m-\mu}{\sigma})\right]\right\} \left\{1 - \exp^{-\gamma e^{\left[-(\frac{m-\mu}{\sigma})\right]}}\right\}^{\alpha-1} dm$$

$$= \int_0^{e^{\left\{-(\frac{y-\mu}{\sigma})\right\}}} \alpha\gamma e^{(-\gamma t)} \left\{1 - e^{(-\gamma t)}\right\}^{\alpha-1} dt \text{ by taking } e^{\left\{-(\frac{m-\mu}{\sigma})\right\}} = t.$$

Putting $e^{(-\gamma t)} = s$ and simplifying, we get

$$S(y) = \left\{1 - e^{-\gamma e^{\left[-(\frac{y-\mu}{\sigma})\right]}}\right\}^{\alpha}. \tag{2.4}$$

The pdf of the standardized random variable $Z = \frac{Y-\mu}{\sigma}$ can be written as

$$\pi(z; \alpha, \gamma) = \alpha\gamma e^{\{-z-\gamma e^{-z}\}}[1 - e^{(-\gamma e^{-z})}]^{\alpha-1}, -\infty < z < \infty, \alpha > 0, \gamma > 0. \tag{2.5}$$

The k^{th} order moment of Z is

$$E(Z^k) = \int_{-\infty}^{\infty} \alpha\gamma z^k e^{\{-z-\gamma e^{-z}\}}[1 - e^{(-\gamma e^{-z})}]^{\alpha-1} dz.$$

By substituting $u = e^{-z}$ and using the following result

$$\int_0^\infty (\log u)^k e^{(-\gamma u)} du = \left.\frac{\partial^k[\gamma^{-a}\Gamma a]}{\partial a^k}\right|_{a=1},$$

we get

$$E(Z^k) = \gamma(-1)^k \Gamma(\alpha+1) \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(\alpha-j)j!} \left\{ \frac{1}{j} \left.\frac{\partial^k[\gamma^{-a}\Gamma a]}{\partial a^k}\right|_{a=1} - \frac{\log(j+1)^k}{\gamma j} \right\}. \tag{2.6}$$

The skewness and kurtosis are given by

$$\text{Skewness} = \frac{\mathbb{E}\left[(Z - \mathbb{E}(Z))^3\right]}{(\mathrm{Var}(Z))^{3/2}}, \quad \text{Kurtosis} = \frac{\mathbb{E}\left[(Z - \mathbb{E}(Z))^4\right]}{(\mathrm{Var}(Z))^2}.$$

Numerical computations of the first four moments can be carried out for selected parameteric values and skewness and kurtosis obtained. The following table gives values of skewness and kurtosis for distribution of Z and also comments about the shape of the distribution.

Table 1: Skewness and Kurtosis for selected parameter values

α	γ	Skewness	Kurtosis	Shape
1.0	0.6	1.28	4.75	Highly right-skewed, heavy-tailed
1.5	1.0	0.89	3.62	Moderately right-skewed
1.9	1.3	0.63	3.18	Mild skewness
2.0	1.8	0.48	3.05	Slightly skewed, near normal
2.3	2.0	0.28	2.91	Almost symmetric, slightly platykurtic

As parameter values increase, the distribution of Z exhibits low skewness and nearly mesokurtic behavior, indicating that it approaches symmetry and resembles the normal distribution.

2.1 Properties of LGIGW distribution

Figure 1 illustrates the density for selected parameter combinations.

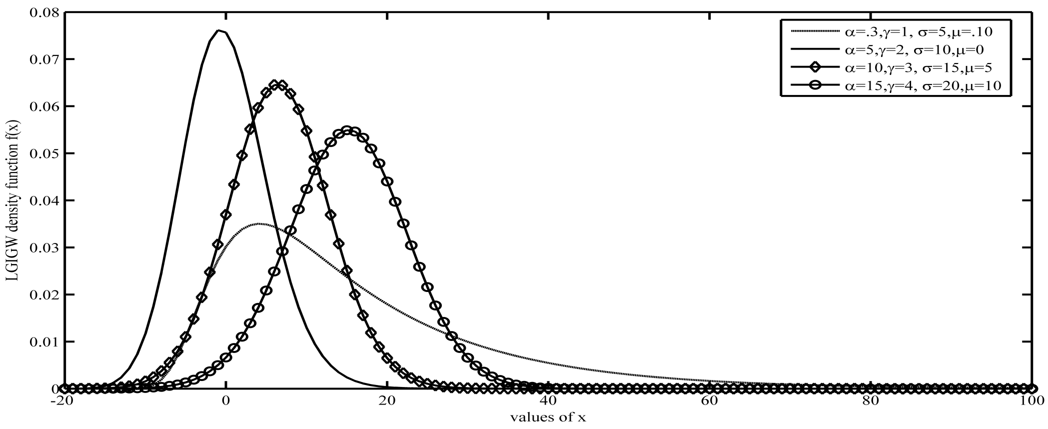

Figure 1: LGIGW density plots

It is observed that

- for small α, the distribution is more right-skewed. An increase in α leads to a reduction in right skewness.
- An increase in γ results in a more pronounced peak and lighter tail of the distribution.
- Higher values of σ increase dispersion while preserving the direction of skewness.
- The plots confirm that LGIGW accommodates a wide range of shapes, including those corresponding to LGIW (α = 1) and LIW ((α = γ = 1)

Remark 1:

- LGIW distribution De Gusmao et al. (2011) is a particular case of LGIGW distribution for $\alpha = 1$.
- For $\alpha = \gamma = 1$,((2.5)) takes the form $e^{\{-z-e^{-z}\}}$, which is the pdf of Inverse Extreme Value distribution or Log Inverse Weibull (LIW) distribution De Gusmao et al. (2011).

In survival studies, some observations are lost to further follow up and are considered to be censored Klein and Moeschberger (2003). In the next section, we discuss LGIGW regression model in case of censoring. The expressions for elements of score vector and information matrix have been derived.

3 Log Generalized Inverse Generalized Weibull Regression Model

In real life, covariates such as age, weight, smoking status and diabetes level affect the lifetimes. If there are n individuals under study, then t_i denotes the lifetime and c_i, the censoring time for i^{th} individual, $i = 1, ..., n$. The lifetimes and the censoring times are assumed to be independent. The response variable y_i is taken to be $\min(\log(t_i), \log(c_i))$. In the regression model, it is assumed that each response variable depends upon p covariates.

A linear regression model for the response variable y_i is written as

$$y_i = \boldsymbol{x_i}^T \boldsymbol{\beta} + \sigma z_i, \;\; i = 1, ..., n \tag{3.1}$$

where

- z_i is the random error with pdf given by (2.5);
- $x_i = (x_{i1}, ..., x_{ip})^T$ is the vector of covariates for $i = 1, ..., n$;
- $\boldsymbol{\beta} = (\beta_1, ..., \beta_p)^T$ is the vector of unknown regression coefficients;
- $\alpha > 0$, $\gamma > 0$ and $\sigma > 0$ are unknown scalar parameters;
- the vector $\boldsymbol{x}_i$ models the location parameter μ_i given by $\boldsymbol{x}_i^T \boldsymbol{\beta}$.

In matrix notation, (3.1) can be written as

$$\boldsymbol{Y} = \boldsymbol{X}^T \boldsymbol{\beta} + \sigma \boldsymbol{Z} \tag{3.2}$$

where

$$\boldsymbol{Y}=\begin{bmatrix} y_1\\ y_2\\ \vdots\\ y_n\end{bmatrix},\quad \boldsymbol{X}=\begin{bmatrix} x_{11} & x_{21} & \dots & x_{n1}\\ x_{12} & x_{22} & \dots & x_{n2}\\ \vdots & \vdots & \ddots & \vdots\\ x_{1p} & x_{2p} & \dots & x_{np}\end{bmatrix},\quad \boldsymbol{\beta}=\left[\beta_1,\beta_2,\because,\beta_p\right],\qquad \boldsymbol{Z}=\left[z_1,z_2,...z_n\right]$$

The survival function of y_i can be estimated as

$$S(y_i,\widehat{\alpha},\widehat{\gamma},\widehat{\sigma},\widehat{\boldsymbol{\beta}}^{\mathbf{T}})=\left[1-e^{\left\{-\widehat{\gamma}e^{(-(\frac{y_i-\boldsymbol{x}_i^T\widehat{\boldsymbol{\beta}}}{\widehat{\sigma}}))}\right\}}\right]^{\widehat{\alpha}}. \tag{3.3}$$

Remark 2:

- For $\alpha=1$, equation (3.1) simplifies to the LGIW regression model introduced by De Gusmao et al. (2011).
- For $\gamma=1$, the proposed model reduces to the Log Inverse Generalized Weibull (LIGW) regression model.
- For $\alpha=1$ and $\gamma=1$, the model further reduces to the Log Inverse Weibull (LIW) regression model.

3.1 Estimation of Parameters

Let

$$F=\{\,i\mid\delta_i=1\,\},\quad C=\{\,i\mid\delta_i=0\,\}$$

denote the sets of observed failures and right-censored observations respectively. Using (2.5) and (3.1), the total log likelihood function for parameters $\boldsymbol{\theta}=(\alpha,\gamma,\sigma,\boldsymbol{\beta}^T)^T$ can be written as

$$\begin{aligned} l(\boldsymbol{\theta})=&\,r\left\{\log\alpha+\log\gamma-\log\sigma\right\}-\sum_{i\in F}(\frac{y_i-\boldsymbol{x}_i{}^T\boldsymbol{\beta}}{\sigma})-\gamma\sum_{i\in F}e^{(-(\frac{y_i-\boldsymbol{x}_i{}^T\boldsymbol{\beta}}{\sigma}))}\\ &+(\alpha-1)\sum_{i\in F}\log\left[1-e^{\{-\gamma e^{(-(\frac{y_i-\boldsymbol{x}_i{}^T\boldsymbol{\beta}}{\sigma}))}\}}\right]+\alpha\sum_{i\in C}\log\left[1-e^{\{-\gamma e^{(-(\frac{y_i-\boldsymbol{x}_i{}^T\boldsymbol{\beta}}{\sigma}))}\}}\right], \end{aligned} \tag{3.4}$$

where r is the number of observed failures.
For simplicity, we write

$$e^{\{-\gamma e^{(-(\frac{y_i-\boldsymbol{x}_i{}^T\boldsymbol{\beta}}{\sigma}))}\}}=m_i \text{ and } (\frac{y_i-\boldsymbol{x}_i{}^T\boldsymbol{\beta}}{\sigma})=z_i \text{ in the sequel.}$$

The MLE of $\boldsymbol{\theta}$ can be obtained by maximizing log-likelihood function given by (3.4).This is attained by solving $\frac{\partial l}{\partial\alpha}=0$, $\frac{\partial l}{\partial\gamma}=0$, $\frac{\partial l}{\partial\sigma}=0$ and $\frac{\partial l}{\partial\beta_j}=0$, where

$$\frac{\partial l}{\partial\alpha}=\frac{r}{\alpha}+\sum_{i\in F}\log\left[1-m_i\right]+\sum_{i\in C}\log\left[1-m_i\right]\ ;$$

$$\frac{\partial l}{\partial\gamma}=\frac{r}{\gamma}-\sum_{i\in F}e^{-z_i}+(\alpha-1)\sum_{i\in F}\frac{m_ie^{-z_i}}{[1-m_i]}+\alpha\sum_{i\in C}\frac{m_ie^{-z_i}}{[1-m_i]}\ ;$$

$$\frac{\partial l}{\partial\sigma}=\frac{-r}{\sigma}+\sum_{i\in F}z_i-\gamma\sum_{i\in F}e^{-z_i}\frac{z_i}{\sigma}+(\alpha-1)\gamma\sum_{i\in F}\frac{m_ie^{-z_i}z_i}{[1-m_i]}+\alpha\gamma\sum_{i\in C}\frac{m_ie^{-z_i}z_i}{[1-m_i]}\ ;$$

$$\frac{\partial l}{\partial\beta_j}=\sum_{i\in F}\frac{x_{ij}}{\sigma}-\gamma\sum_{i\in F}e^{-z_i}\frac{x_{ij}}{\sigma}+(\alpha-1)\gamma\sum_{i\in F}\frac{m_ie^{-z_i}\frac{x_{ij}}{\sigma}}{[1-m_i]}+\alpha\gamma\sum_{i\in C}\frac{m_ie^{-z_i}\frac{x_{ij}}{\sigma}}{[1-m_i]}\ .$$

As it is difficult to find the MLEs analytically, hence we shall be using numerical methods for finding the estimates in Section 4.

For interval estimation and testing of hypotheses for the parameters, the 4×4 observed information matrix is obtained as

$$\boldsymbol{J} = \boldsymbol{J}\left(\boldsymbol{\theta}\right) = -\begin{bmatrix} \mathrm{J}_{\alpha,\alpha} & \mathrm{J}_{\alpha,\gamma} & \mathrm{J}_{\alpha,\sigma} & \mathrm{J}_{\alpha,\beta_j} \\ & \mathrm{J}_{\gamma,\gamma} & \mathrm{J}_{\gamma,\sigma} & \mathrm{J}_{\gamma,\beta_j} \\ & & \mathrm{J}_{\sigma,\sigma} & \mathrm{J}_{\sigma,\beta_j} \\ & & & \mathrm{J}_{\beta_j,\beta_s} \end{bmatrix}.$$

The expression for elements of matrix J have been included in the Appendix.
The parameter space is defined such that all scale and shape parameters are strictly positive, while location-type parameters are real-valued, and the true parameter lies in the interior of the parameter space. The LGIGW model is identifiable, as distinct parameter values correspond to distinct distributions due to the identifiability of its component distributions and the one-to-one transformation structure. The log-likelihood function is assumed to be twice continuously differentiable, and the Fisher information matrix is finite and positive definite. In the presence of censored observations, a non-informative and independent censoring mechanism is assumed, ensuring the validity of the likelihood function. Under these regularity conditions, the maximum likelihood estimator exists, is consistent, and asymptotically normal.

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} \mathcal{N}_{p+3}\left(0, I(\theta)^{-1}\right), \tag{3.5}$$

where $I(\theta)$ is the expected information matrix, approximated by the observed information matrix $J(\hat{\theta})$. Thus,

$$\hat{\theta} \approx \mathcal{N}_{p+3}\left(\theta, J(\hat{\theta})^{-1}\right). \tag{3.6}$$

An approximate $100(1-\xi)\%$ confidence interval for parameter θ_j is

$$\hat{\theta}_j \pm z_{\xi/2}\sqrt{\widehat{\mathrm{Var}}(\hat{\theta}_j)}, \tag{3.7}$$

where

- $z_{\xi/2}$ is the standard normal quantile,
- $\widehat{\mathrm{Var}}(\hat{\theta}_j)$ is the j^{th} diagonal element of $J(\hat{\theta})^{-1}$.

These intervals allow statistical inference for regression coefficients and shape parameters.

4 Simulation Study

4.1 Design of the Study

A Monte Carlo simulation study was conducted to evaluate the finite-sample performance of the maximum likelihood estimators (MLEs) of the LGIGW regression model under Type I censoring.
Data were generated from model (3.1) with two covariates. The true parameter values were chosen as $\alpha = 0.50$, $\gamma = 1.50, \quad \sigma = 0.80, \quad \beta = (0.50, -0.30)^T$.
Covariates were generated as $x_{i1} \sim$ Uniform$(0,1)$, $\quad x_{i2} \sim$ Uniform$(0,1)$.
Sample sizes considered were $n = 30, 50, 100$.
Type I censoring was imposed to achieve approximately 20% and 40% censoring rates.
For each configuration, 1000 replications were performed.

4.2 Performance Measures

The following metrics were computed:

- Bias: $E(\hat{\theta} - \theta)$
- Mean Squared Error (MSE): $E[(\hat{\theta} - \theta)^2]$
- Coverage Probability (CP) of 95% confidence intervals

The model parameters were estimated via numerical maximization of the log-likelihood function using a quasi-Newton algorithm in R software. Convergence was declared when the change in parameter estimates and the log-likelihood values between successive iterations fell below 10^{-6} and 10^{-8} respectively, or when the maximum number of iterations (1000) was reached. The optimization routine was implemented in R using the optim function, and convergence was assessed based on both relative tolerance and gradient norm criteria.

4.3 Results

Tables 2 and 3 present results under 20% and 40% censoring respectively.

Table 2: Simulation Results under 20% Censoring

Parameter	n	Bias	MSE	CP (95%)
α	30	0.021	0.018	0.93
	50	0.012	0.010	0.94
	100	0.005	0.004	0.95
γ	30	0.084	0.126	0.92
	50	0.042	0.068	0.94
	100	0.018	0.022	0.95
σ	30	0.016	0.012	0.94
	50	0.008	0.006	0.95
	100	0.003	0.002	0.95
β_1	30	0.014	0.008	0.94
	50	0.006	0.004	0.95
	100	0.002	0.001	0.95
β_2	30	-0.018	0.009	0.93
	50	-0.007	0.004	0.94
	100	-0.003	0.002	0.95

Table 3: Simulation Results under 40% Censoring

Parameter	n	Bias	MSE	CP (95%)
α	30	0.038	0.031	0.91
	50	0.020	0.018	0.93
	100	0.009	0.007	0.94
γ	30	0.131	0.218	0.89
	50	0.064	0.108	0.92
	100	0.025	0.042	0.94
σ	30	0.028	0.021	0.92
	50	0.013	0.010	0.94
	100	0.006	0.004	0.95
β_1	30	0.021	0.014	0.92
	50	0.009	0.006	0.94
	100	0.004	0.002	0.95
β_2	30	-0.026	0.015	0.90
	50	-0.012	0.007	0.93
	100	-0.005	0.003	0.94

The simulation results indicate that

- the MLEs are approximately unbiased.
- Bias and MSE decrease as sample size increases.
- Coverage probabilities approach the nominal 95% level for $n \geq 50$.
- Higher censoring increases variability, particularly for γ, but does not induce systematic bias.

5 Applications

Application of LGIGW regression model has been explored for two cases
(i) n =15 and
(ii) n =40.
(i) The bone marrow transplant censored data reported in Klein and Moeschberger (2003) are used to illustrate the applicability of the LGIGW regression model. The data set consists of survival times (in days) to death or relapse for 15 patients diagnosed with non-Hodgkin lymphoma (NHL) who received allogeneic (Allo) bone marrow transplants from HLA-matched sibling donors. The primary objective is to assess the effect of covariates on the logarithm of the survival times.

We fit the LGIGW regression model without intercept

$$y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \sigma z_i, \quad i = 1, ..., 15,$$

where the errors $z_1, ..., z_{15}$ are independent random variables with density function given in (2.5).

MLEs of the parameters with their Standard Errors (SEs) are reported in Table 4 for LGIGW, LGIW and LIW regression models fitted to the data set. The p-values have also been reported for testing the significance of regression coefficients.

Table 4: Estimates of the parameters and standard errors (SEs)

Model	estimates	SE	p-value
LGIGW	$\widehat{\alpha} = .1603$	.0044	-
	$\widehat{\gamma} = 1.2466$	.7727	-
	$\widehat{\sigma} = .5434$	.0270	-
	$\widehat{\beta_1} = .0581$	.0001	.000
	$\widehat{\beta_2} = -.0236$	.0002	.000
LGIW	$\widehat{\alpha} = 1.0000$	-	-
	$\widehat{\gamma} = 3.1157$	9.9616	-
	$\widehat{\sigma} = .4111$	.0076	-
	$\widehat{\beta_1} = .0546$	.0000	.000
	$\widehat{\beta_2} = -.0218$	.0001	.000
LIW	$\widehat{\alpha} = 1$	-	-
	$\widehat{\gamma} = 1$	-	-
	$\widehat{\sigma} = .4602$	.0117	-
	$\widehat{\beta_1} = .0598$	.0000	.000
	$\widehat{\beta_2} = -.0153$	.0000	.000

The results presented in Table 4 provide important insights into the performance of the LGIGW regression model in comparison with its submodels. It is observed that the regression coefficients β_1 and β_2 are statistically significant at the 5% level across all models, indicating that the selected covariates have a significant impact on the logarithm of survival times. The parameter estimates under the LGIGW model appear stable with relatively small standard errors, suggesting reliable estimation. In particular, the additional shape parameter α in the LGIGW model enhances flexibility, allowing the model to capture more complex data structures as compared to the LGIW and LIW models.
The sample size (n = 15) is relatively small. Therefore, inference has to be interpreted cautiously. The asymptotic normal approximation may be less accurate in such small samples. However, the analysis is presented primarily to illustrate model applicability rather than to draw definitive clinical conclusions.
LGIGW regression model has been compared with LGIW and LIW regression models and AIC, BIC and AICc values for these models are provided in Table 5.

Table 5: AIC, BIC and AICc for LGIGW, LGIW and LIW regression models

Model	AIC	BIC	AICc
LGIGW	46.232	50.406	44.865
LGIW	112.466	115.299	110.133
LIW	92.904	95.028	90.504

It is seen that the values of AIC, BIC and AICc are lowest for LGIGW regression model which establishes the superiority of our regression model.

(ii) A dataset consisting of 40 financial loan records was analyzed, where the response variable represents time to default in months. Right censoring corresponds to loans that remained active at the end of the study. Two covariates were considered: normalized credit score and borrower risk category. The dataset exhibits skewness and heavy-tailed behavior, making it suitable for flexible parametric modeling. We fit the LGIGW regression model with intercept

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \sigma z_i, \quad i = 1, ..., 40,$$

where the errors $z_1, ..., z_{40}$ are independent random variables with density function given in (2.5).
The LGIGW regression model provided the best fit in terms of AIC and BIC when compared with its submodels. Dataframe is
time(t_i) = (4, 6, 7, 9, 11, 14, 18, 22, 5, 8, 10, 13, 17, 21, 26, 30, 6, 9, 12, 15, 19, 23, 28, 35, 7, 11, 14, 18, 24, 32, 40, 50, 8, 12, 16, 20, 27, 36, 45, 60),
status (δ_i) = (1, 1, 1, 1, 1, 1, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0),
x_1 (Credit Score) = (0.40, 0.50, 0.45, 0.55, 0.60, 0.62, 0.75, 0.80, 0.42, 0.48, 0.53, 0.58, 0.72, 0.78, 0.85, 0.88, 0.44, 0.49, 0.57, 0.61, 0.74, 0.79, 0.86, 0.90, 0.46, 0.52, 0.59, 0.73, 0.81, 0.87, 0.92, 0.95, 0.47, 0.54, 0.60, 0.76, 0.84, 0.91, 0.94, 0.97),
x_2 (Risk) = (1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0)

Maximum likelihood estimates, standard errors, and p-values for LGIGW, LGIW, LIW, and Weibull regression models are presented in Table 6.

Table 6: Maximum likelihood estimates, standard errors, and p-values for LGIGW, LGIW, LIW, and Weibull regression models

Model	Parameter	Estimate	Std. Error	p-value
LGIGW	β_0	1.245	0.312	0.0002
	β_1	-2.138	0.845	0.0115
	β_2	0.965	0.402	0.0189
	α	1.872	0.556	0.0011
	λ	0.743	0.221	0.0009
	γ	1.356	0.478	0.0047
LGIW	β_0	1.102	0.295	0.0005
	β_1	-1.845	0.801	0.0190
	β_2	0.812	0.376	0.0302
	α	1.645	0.498	0.0023
	γ	1.210	0.442	0.0068
LIW	β_0	0.954	0.270	0.0012
	β_1	-1.502	0.756	0.0450
	β_2	0.690	0.350	0.0485
	γ	1.085	0.410	0.0095
Weibull	β_0	0.875	0.260	0.0015
	β_1	-1.320	0.710	0.0620
	β_2	0.580	0.330	0.0785
	Shape	1.950	0.520	0.0008

Note: Significant parameters (p-value < 0.05) indicate important covariate effects. The LGIGW model shows more statistically significant parameters, reflecting its flexibility and better fit.

The following table compares LGIGW, LGIW, LIW and Weibull regression models on basis of goodness-of-fit criteria.

Table 7: Comparison of LGIGW, LGIW, LIW and Weibull regression models based on goodness-of-fit criteria

Model	Log-Likelihood	AIC	BIC
LGIGW	13.88	**-15.769**	**-7.786**
LGIW	4.449	1.101	9.546
LIW	11.207	-14.415	-7.659
Weibull	-30.532	69.064	75.819

Among the competing models, the LGIGW regression model provides the lowest AIC and BIC values, indicating superior goodness-of-fit. The Weibull model shows comparatively poorer performance, suggesting that it fails to capture the underlying heavy-tailed and skewed behaviour of the data. Since outliers in any data set can vary the results of analysis, hence an important step in the analysis of data is to detect the outliers and influential observations using sensitivity and residual analysis. For this, several approaches exist in literature given by Chatterjee and Hadi (1988); Cook (1977, 1986); Weisberg and Cook (1982).
In the following section, we briefly outline the methods used for sensitivity analysis and for deriving the necessary mathematical expressions.

6 Sensitivity Analysis

Different approaches for detecting the outliers are discussed in the following subsections.

6.1 Global Influence

The impact of removing the i^{th} observation from the data set is assessed through the case-deletion approach, a widely used method in global influence analysis (Cook, 1977). The case-deletion model corresponding to model (3.1) is given by

$$y_j = \boldsymbol{x_j}^T\boldsymbol{\beta} + \sigma z_j, \;\; j = 1, ..., n \text{ and } j \neq i. \tag{6.1}$$

The subscript 'i' indicates the original quantity but with deletion of i^{th} observation. If the estimates are seriously affected by deleting an observation, then that observation needs attention.
If $\widehat{\boldsymbol{\theta}}_{(i)}$: estimate of parameters obtained by deleting i^{th} observation;
$\widehat{\boldsymbol{\theta}}$: original estimate of parameters; then to measure the global influence,
Generalized Cook Distance is given by

$$GD_i(\boldsymbol{\theta}) = (\widehat{\boldsymbol{\theta}}_{(i)} - \widehat{\boldsymbol{\theta}})^T \boldsymbol{J}(\widehat{\boldsymbol{\theta}})^{-1}(\widehat{\boldsymbol{\theta}}_{(i)} - \widehat{\boldsymbol{\theta}}), \tag{6.2}$$

Another measure to assess the difference between $\widehat{\boldsymbol{\theta}}_{(i)}$ and $\widehat{\boldsymbol{\theta}}$ is the likelihood distance given by

$$LD_i(\theta) = 2\{l(\widehat{\boldsymbol{\theta}}) - l(\widehat{\boldsymbol{\theta}}_{(i)})\}, \tag{6.3}$$

where
$l(\widehat{\boldsymbol{\theta}})$: original log-likelihood function;
$l(\widehat{\boldsymbol{\theta}}_{(i)})$: log-likelihood function obtained after deleting i^{th} observation.
If $\widehat{\boldsymbol{\theta}}_{(i)}$ is far from $\widehat{\boldsymbol{\theta}}$, then i^{th} observation can be considered as an influential observation.

The values of $GD_i(\boldsymbol{\theta})$ and $LD_i(\boldsymbol{\theta})$ are given in Table 8.

Table 8: Generalized Cook's Distances and Likelihood Distances

i	$GD_i(\boldsymbol{\theta})$	$LD_i(\boldsymbol{\theta})$
1	.1259	-6.8353
2	13.1495	170.4717
3	.7788	-2.7328
4	.2601	-2.1485
5	13.9921	4.5275
6	.0020	-3.5609
7	.0187	-3.6358
8	.0592	.9726
9	.0591	1.3945
10	.0592	.8294
11	.0592	.6291
12	.0107	-.2799
13	.2870	-.7998
14	.0011	-1.5854
15	.0286	.4901

The global influence measures as given in Table 8 clearly identify observations 2 and 5 as influential, as they exhibit substantially larger Generalized Cook's distances compared to other observations. The likelihood distance values further confirm this finding. The index plots of influence measures are shown in Figures 2 and 3.

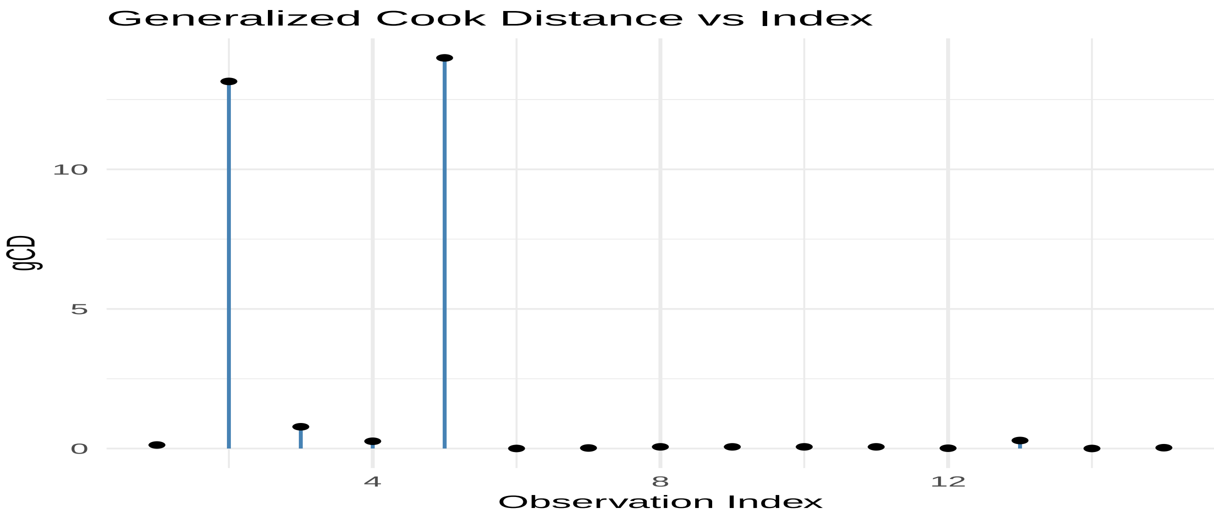

Figure 2: Generalized Cook's Distance

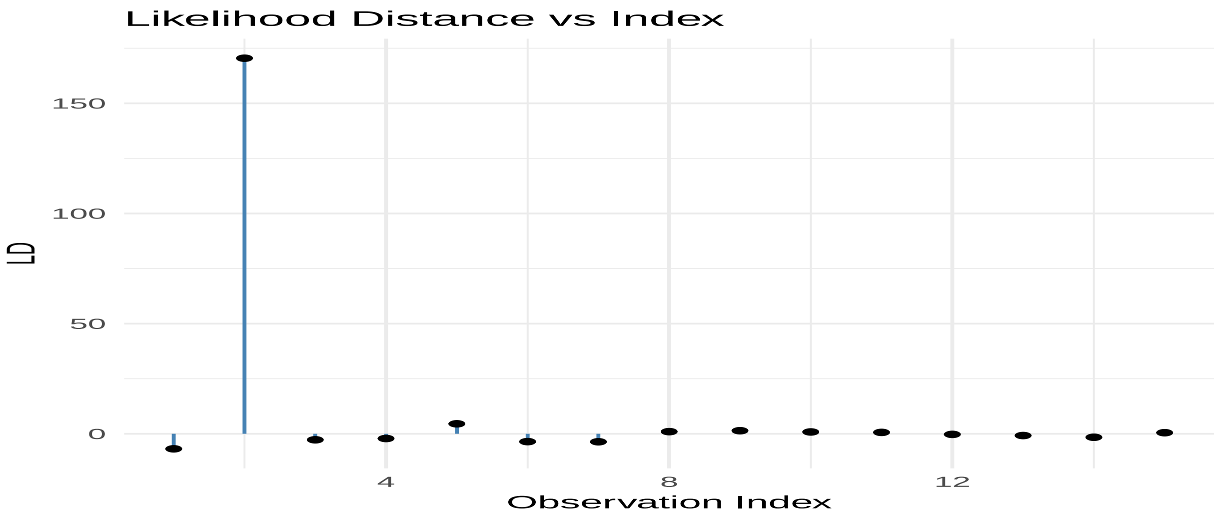

Figure 3: Likelihood Distance

The above figures also help in concluding that observations 2 and 5 are the influential observations.

6.2 Local Influence

An alternative approach to sensitivity analysis is the local influence method introduced by Cook (1986). This method involves assigning weights to observations rather than deleting them, and assessing their impact on the parameter estimates. The resulting framework is commonly referred to as a perturbed model. If $\widehat{\boldsymbol{\theta}}_w$ denotes the vector of estimates under perturbed model, then $LD(w) = 2\{l(\widehat{\boldsymbol{\theta}}) - l(\widehat{\boldsymbol{\theta}}_w)\}$ is the likelihood displacement. The normal curvature at direction $\boldsymbol{d}$ is given by

$$\boldsymbol{C_d}(\boldsymbol{\theta}) = 2|\boldsymbol{d}^T\boldsymbol{\Delta}^T J(\widehat{\boldsymbol{\theta}})^{-1}\boldsymbol{\Delta d}| \tag{6.4}$$

such that $||\boldsymbol{d}|| = 1$ and $\boldsymbol{\Delta}$ is a (p+3)×n matrix. We can also calculate $\boldsymbol{C_d}(\alpha), \boldsymbol{C_d}(\gamma), \boldsymbol{C_d}(\sigma)$ and $\boldsymbol{C_d}(\beta)$. The elements of $\boldsymbol{\Delta}$ matrix are calculated under various perturbation schemes explained below. The largest eigen value of the matrix $\boldsymbol{B} = \boldsymbol{\Delta}^T\boldsymbol{J}(\widehat{\boldsymbol{\theta}})^{-1}\boldsymbol{\Delta}$ is $C_{d_{max}}$ where $\boldsymbol{d}_{max}$ is the eigen vector corresponding to $C_{d_{max}}$. The index plots of $\boldsymbol{d}_{max}, \boldsymbol{C_{d_i}}(\alpha), \boldsymbol{C_{d_i}}(\gamma), \boldsymbol{C_{d_i}}(\sigma)$ and $\boldsymbol{C_{d_i}}(\boldsymbol{\beta})$ together denote the total local influence Lesaffre and Verbeke (1998). Hence the curvature has the form $C_i = 2|\boldsymbol{\Delta}_i^T J(\widehat{\boldsymbol{\theta}})^{-1}\boldsymbol{\Delta}_i|$ at direction $\boldsymbol{d}$. For the vector of weights $\boldsymbol{w} = (w_1, ..., w_n)^T$, let the log-likelihood (3.4), under perturbed scheme be $l(\boldsymbol{\theta}|\boldsymbol{w})$. Then

$$\boldsymbol{\Delta} = (\boldsymbol{\Delta}_{ji})_{(p+3)\times n} = \frac{\partial^2 l(\boldsymbol{\theta}|\boldsymbol{w})}{\partial\theta_j\partial w_i}, i = 1, ..., n,\ j = 1, ..., p+3 \tag{6.5}$$

is calculated under different perturbed schemes.

- **Case-weight Perturbation**: It is the basis for the study of influence. To perturb the contribution of each case, the vector of weights $\boldsymbol{w}$ is used where w_j=1 (inclusion) or w_j=0 (exclusion) of an observation from the estimation of $\boldsymbol{\theta}$. This method enables us to check the relative importance of observations in estimation.
  Under this scheme, the log-likelihood has the form

$$\begin{aligned} l(\boldsymbol{\theta}|\boldsymbol{w}) = \{\log\alpha + \log\gamma - \log\sigma\}\sum_{i\in F} w_i - \sum_{i\in F} w_i z_i - \gamma\sum_{i\in F} w_i e^{-z_i} \\ + (\alpha-1)\sum_{i\in F} w_i \log\left[1 - e^{\{-\gamma e^{-z_i}\}}\right] + \alpha\sum_{i\in C} w_i\log\left[1 - e^{\{-\gamma e^{-z_i}\}}\right], \end{aligned} \tag{6.6}$$

  where $z_i = (\frac{y_i - \boldsymbol{x}_i{}^T\boldsymbol{\beta}}{\sigma})$ and $0 \leq w_i \leq 1$.

  For $\widehat{z}_i = (\frac{y_i - \boldsymbol{x}_i{}^T\widehat{\boldsymbol{\beta}}}{\widehat{\sigma}})$, $\widehat{m}_i = e^{\{-\widehat{\gamma}e^{-\widehat{z}_i}\}}$, the elements of $\boldsymbol{\Delta}$ matrix are given in the Appendix.

- **Response Variable Perturbation**:
Each y_i is perturbed as $y_{iw} = y_i + w_i S_y$, where S_y is standard deviation of elements contained in $\boldsymbol{Y}$ and $w_i \in \Re$. Under this scheme, the log-likelihood has the form

$$\begin{aligned} l(\boldsymbol{\theta}|\boldsymbol{w}) &= r\{\log\alpha + \log\gamma - \log\sigma\} - \sum_{i\in F} z_i^* - \gamma\sum_{i\in F} e^{-z_i^*} \\ &\quad + (\alpha-1)\sum_{i\in F}\log\left[1 - e^{\{-\gamma e^{-z_i^*}\}}\right] + \alpha\sum_{i\in C}\log\left[1 - e^{\{-\gamma e^{-z_i^*}\}}\right], \end{aligned} \tag{6.7}$$

where $z_i^* = \dfrac{(y_i + w_i S_y) - \boldsymbol{x_i}^T\boldsymbol{\beta}}{\sigma}$.
For $\widehat{z_i}^* = \dfrac{(y_i + w_i S_y) - \boldsymbol{x_i}^T\widehat{\boldsymbol{\beta}}}{\widehat{\sigma}}$, and $\widehat{m_i}^* = e^{\{-\widehat{\gamma} e^{-\widehat{z_i}^*}\}}$, the elements of $\boldsymbol{\Delta}$ matrix are given in the Appendix.

- **Explanatory Variable Perturbation**:
On a particular explanatory variable, say X_t, we apply an additive perturbation scheme by writing $x_{itw} = x_{it} + w_i S_x$, where S_x is the estimated standard deviation of perturbed explanatory variable and $w_i \in \Re$. The perturbed log-likelihood has the form

$$\begin{aligned} l(\boldsymbol{\theta}|\boldsymbol{w}) &= r\{\log\alpha + \log\gamma - \log\sigma\} - \sum_{i\in F} z_i^{**} - \gamma\sum_{i\in F} e^{-z_i^{**}} \\ &\quad + (\alpha-1)\sum_{i\in F}\log\left[1 - e^{\{-\gamma e^{-z_i^{**}}\}}\right] + \alpha\sum_{i\in C}\log\left[1 - \exp\{-\gamma e^{-z_i^{**}}\}\right], \end{aligned} \tag{6.8}$$

where $z_i^{**} = \dfrac{y_i - \boldsymbol{x}_i^{*T}\boldsymbol{\beta}}{\sigma}$ and $\boldsymbol{x}_i^{*T}\boldsymbol{\beta} = \beta_1 + \beta_2 x_{i2} + ... + \beta_t(x_{it} + w_i S_x) + ... + \beta_p x_{ip}$.
The elements of $\boldsymbol{\Delta}$ matrix are included in the Appendix.

By applying case weight perturbation for considered data set using LGIGW regresssion model, we obtain the value of $C_{d_{max}} = 11.823$. The plots of eight perturbation ,eigen vector corresponding to d_{max} are shown in Figure 4 (a) and total influence C_i are shown in Figure 4 (b).

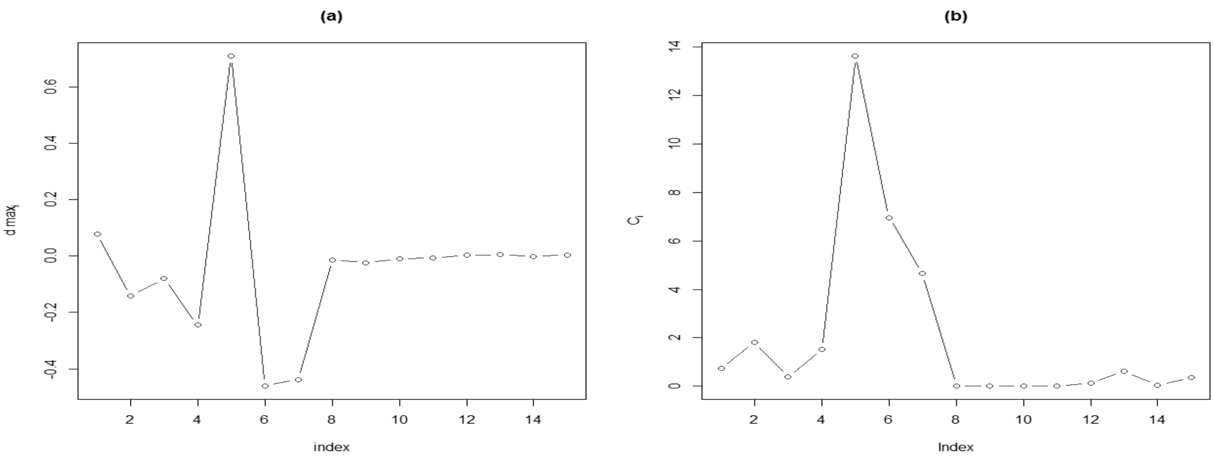

Figure 4: Index plots of (a) $d_{max_i}(\boldsymbol{\theta})$ and (b) total local influence $C_i(\boldsymbol{\theta})$ for case-weight perturbation

From Figures 4 (a) and 4 (b), it can be concluded that observations 2 and 5 are the influential observations. Using Response variable perturbation, maximum curvature $C_{d_{max}}$ is obtained as 3.971.

Figure 5(a) gives the plot of d_{max} and it is observed that the observations 2 and 5 are far away from other observations. Total local influence C_i versus observation index has been plotted in Figure 5(b) versus observation index.

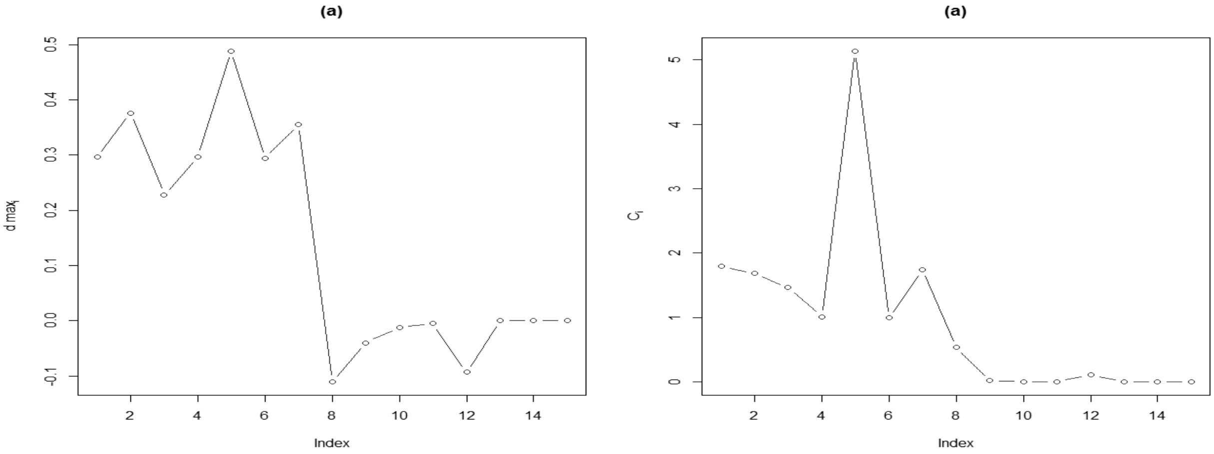

Figure 5: Index plots of (a) $d_{max_i}(\boldsymbol{\theta})$ and (b) total local influence $C_i(\boldsymbol{\theta})$ under response variable perturbation

Under local influence analysis, the maximum curvature values obtained from case-weight and response perturbation schemes indicate that observations 2 and 5 contribute disproportionately to parameter instability. However, subsequent refitting after removing observation 5 reveals only moderate relative changes in parameter estimates, indicating that the overall inference of the model remains stable. Thus, although influential, the detected observations do not materially alter substantive conclusions

7 Residual analysis

After fitting a model to the data, residual analysis (Collett (2015); Ortega et al. (2008)) is carried out to check the assumptions of the model, validity of the data, study the departures from error assumption and identify the outliers. Martingale residual was proposed by Barlow and Prentice (1988) and studied by Fleming and Harrington (2011). Therneau et al. (1990) introduced deviance residual for the Cox model with no time-dependent covariates. The two types of residuals are detailed in the following subsections.

7.1 Martingale residual

The residual for LGIGW regression model can be written as

$$r_{M_i} = \delta_i + \log\left(S(y_i; \widehat{\alpha}, \widehat{\gamma}, \widehat{\sigma}, \widehat{\boldsymbol{\beta}}^{\mathbf{T}})\right), \tag{7.1}$$

where

$$\delta_i = \begin{bmatrix} 1 & \text{if } i \in F, \\ 0 & \text{if } i \in C. \end{bmatrix}$$

For $\widehat{z}_i = (\dfrac{y_i - \boldsymbol{x}_i^T \widehat{\boldsymbol{\beta}}}{\widehat{\sigma}})$,

$$r_{M_i} = \begin{bmatrix} 1 + \widehat{\alpha}\left[1 - \exp\left\{-\widehat{\gamma}\exp\left(-\widehat{z}_i\right)\right\}\right] & \text{if } i \in F, \\ \widehat{\alpha}\left[1 - \exp\left\{-\widehat{\gamma}\exp\left(-\widehat{z}_i\right)\right\}\right] & \text{if } i \in C. \end{bmatrix} \tag{7.2}$$

The maximum value of residual r_{M_i} is +1 and minimum is $-\infty$.

7.2 Modified deviance residual (martingale-type residual)

The transformation of martingale residual based on deviance has been used on LGIGW regression model to get a new residual, distributed symmetrically around zero. The modified deviance residual for LGIGW regression model can be written as

$$r_{D_i} = sgn(r_{M_i}) {-2}[r_{M_i} + \delta_i \log\left(\delta_i - r_{M_i}\right)]^{1/2} = \begin{bmatrix} sgn(r_{M_i})\sqrt{-2[r_{M_i} + \log\left(1 - r_{M_i}\right)]} & \text{if } i \in F, \\ sgn(r_{M_i})\sqrt{-2 r_{M_i}} & \text{if } i \in C. \end{bmatrix} \tag{7.3}$$

where r_{M_i} is the martingale residual.

Martingale-type residuals r_{D_i} versus fitted values have been plotted in Figure 6 for detection of the outliers.

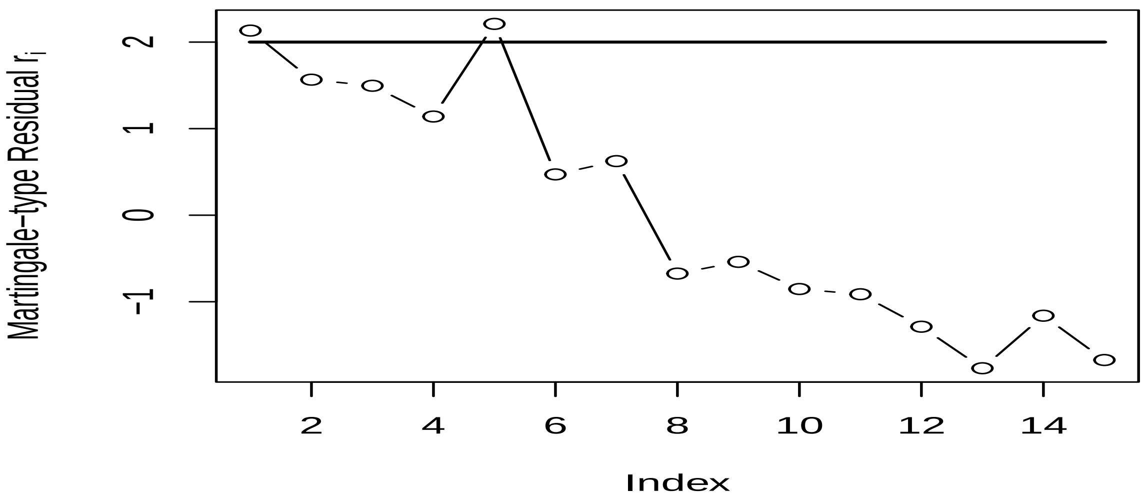

Figure 6: Plot for martingale-type residual

Apart from observations 1 and 5, all observations are lying in [-2,2], so we can conclude that model is fitting well to considered data set.

7.3 Impact of detected outliers

The relative change in estimate of a parameter can be calculated by using the formula

$$RC_{\boldsymbol{\theta}_j} = [(\widehat{\boldsymbol{\theta_j}} - \widehat{\boldsymbol{\theta}}_j(I))/\widehat{\boldsymbol{\theta_j}}] \tag{7.4}$$

where I: set of influential observations;
and $\widehat{\boldsymbol{\theta}}_j(I)$: the estimates of parameters obtained after removal of influential observations.
From sensitivity and residual analysis, we observe that observation 5 is the possible outlier. To evaluate the influence of this observation, the model was re-estimated after excluding it from the original data set. Table 9 reports the parameter estimates for both the complete data and the reduced data set obtained after removing the outlier. In addition, the relative percentage changes in the parameter estimates and their corresponding p-values are presented.

Table 9: Parameter estimates, relative changes in parameters (in %) and p-values for regression coefficients

Observations		$\widehat{\boldsymbol{\alpha}}$	$\widehat{\boldsymbol{\gamma}}$	$\widehat{\boldsymbol{\sigma}}$	$\widehat{\boldsymbol{\beta}}_1$	$\widehat{\boldsymbol{\beta}}_2$
All	estimate	.16034	1.2466	.5434	.0581	-.0236
	RC (%)	-	-	-	-	-
	p-value	-	-	-	.000	.000
All-[#5]	estimate	.1167	5.6581	.4962	.0569	-.0317
	RC(%)	27.245	-353.885	8.691	2.179	-34.349
	p-value	-	-	-	.000	.000

The results presented in Table 9 indicate that the parameter estimates of the LGIGW regression model are not highly sensitive to the deletion of the 5^{th} observation. This is evident from the fact that the removal of this potentially influential observation does not alter the statistical significance of the parameter estimates at the 5% level. Consequently, the overall inference drawn from the model remains unchanged for the data set under consideration. Table 8 further compares the parameter estimates obtained before and after the removal of the 5^{th} observation. Although the shape parameter γ exhibits a relatively large percentage change, the regression coefficients β_1 and β_2 show only moderate changes of 2.18% and 34.35%, respectively. Importantly, both coefficients remain statistically significant at the 5% level. It is worth noting that, while the present study demonstrates that the removal of the outlier does not substantially affect the model estimates for this particular data set. Different data sets especially those involving censored observations may exhibit greater sensitivity to such deletions. Thus, the analysis primarily illustrates the methodology for detecting influential observations and assessing relative percentage changes.

8 Conclusions

This paper introduced the Log Generalized Inverse Generalized Weibull (LGIGW) distribution and its associated regression model for censored lifetime data. The proposed model generalizes several well-known models including LGIW and LIW regression models, thereby offering a unified modeling framework. Theoretical properties of the distribution were derived, and maximum likelihood estimation was developed under Type I censoring. Asymptotic inference procedures, including confidence interval construction, were established using the observed information matrix. Through applications to a bone marrow transplant and financial data sets, the LGIGW regression model demonstrated superior performance compared to its submodels, as evidenced by lower AIC, BIC, and AICc values. Sensitivity and residual diagnostics revealed two influential observations. However, their removal did not materially affect the statistical significance of the regression coefficients, demonstrating robustness of the proposed model. Overall, the LGIGW regression model provides a flexible and powerful alternative for modeling censored lifetime data. Future research may consider Bayesian estimation, Progressive censoring schemes, Interval-censored extensions and Multivariate generalizations.

Declarations

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Consent to Publish: Consent to publish has been obtained.
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Appendix

The expressions of elements of matrix $\boldsymbol{J}$ are given below:

$$\mathrm{J}_{\alpha,\alpha} = \frac{\partial^2 \mathrm{l}}{\partial \alpha^2} = \frac{-r}{\alpha^2};$$

$$\mathrm{J}_{\alpha,\gamma} = \frac{\partial^2 \mathrm{l}}{\partial \alpha \partial \gamma} = \sum_{i \in F} \frac{m_i e^{-z_i}}{[1-m_i]} + \sum_{i \in C} \frac{m_i e^{-z_i}}{[1-m_i]};$$

$$\mathrm{J}_{\alpha,\sigma} = \frac{\partial^2 \mathrm{l}}{\partial \alpha \partial \sigma} = \gamma \sum_{i \in F} \frac{m_i e^{-z_i} (\frac{z_i}{\sigma})}{[1-m_i]} + \gamma \sum_{i \in C} \frac{m_i e^{-z_i} (\frac{z_i}{\sigma})}{[1-m_i]};$$

$$\begin{aligned}
\mathrm{J}_{\sigma,\sigma} = \frac{\partial^2 \mathrm{l}}{\partial \sigma^2} &= \frac{r}{\sigma^2} - 2\sum_{i \in F} \frac{z_i}{\sigma^2} + \gamma \sum_{i \in F} e^{-z_i} \left[-(\frac{z_i}{\sigma})^2 + 2\frac{z_i}{\sigma^2} \right] \\
&+ (\alpha - 1) \sum_{i \in F} \left[\frac{-\gamma^2 m_i^2 e^{(-2z_i)} (\frac{z_i}{\sigma})^2}{[1-m_i]^2} - \frac{\gamma m_i e^{-z_i} [2\frac{z_i}{\sigma^2} - (\frac{z_i}{\sigma})^2 + \gamma e^{-z_i} (\frac{z_i}{\sigma})^2]}{[1-m_i]} \right] \\
&+ \alpha \sum_{i \in C} \left[\frac{-\gamma^2 m_i^2 e^{-2z_i} (\frac{z_i}{\sigma})^2}{[1-m_i]^2} - \frac{\gamma m_i e^{-z_i} [2\frac{z_i}{\sigma^2} - (\frac{z_i}{\sigma})^2 + \gamma e^{-z_i} (\frac{z_i}{\sigma})^2]}{[1-m_i]} \right];
\end{aligned}$$

$$\mathrm{J}_{\gamma,\gamma} = \frac{\partial^2 \mathrm{l}}{\partial \gamma^2} = \frac{-r}{\gamma^2} - (\alpha - 1) \sum_{i \in F} \frac{m_i e^{-2z_i}}{[1-m_i]^2} - \alpha \sum_{i \in C} \frac{m_i e^{-2z_i}}{[1-m_i]^2};$$

$$\begin{aligned}
\mathrm{J}_{\gamma,\sigma} = \frac{\partial^2 \mathrm{l}}{\partial \gamma \partial \sigma} &= \sum_{i \in F} e^{-z_i} (\frac{-z_i}{\sigma}) + (\alpha - 1) \sum_{i \in F} \left[\frac{m_i^2 e^{-2z_i} \gamma (\frac{-z_i}{\sigma})}{[1-m_i]^2} + \frac{m_i \exp -z_i (\frac{-z_i}{\sigma}) [1 - \gamma e^{-z_i}]}{[1-m_i]} \right] \\
&+ \alpha \sum_{i \in C} \left[\frac{m_i^2 \exp -2z_i \gamma (\frac{-z_i}{\sigma})}{[1-m_i]^2} + \frac{m_i e^{-z_i} (\frac{-z_i}{\sigma}) [1 - \gamma e^{-z_i}]}{[1-m_i]} \right].
\end{aligned}$$

For $j = 1, 2, ..., p$

$$\mathrm{J}_{\alpha,\beta_j} = \frac{\partial^2 \mathrm{l}}{\partial \alpha \partial \beta_{\mathrm{j}}} = \gamma \sum_{i \in F} \frac{m_i e^{-z_i} (\frac{x_{ij}}{\sigma})}{[1-m_i]} + \gamma \sum_{i \in C} \frac{m_i e^{-z_i} (\frac{x_{ij}}{\sigma})}{[1-m_i]};$$

$$\begin{aligned}
\mathrm{J}_{\gamma,\beta_j} = \frac{\partial^2 \mathrm{l}}{\partial \gamma \partial \beta_{\mathrm{j}}} &= \sum_{i \in F} e^{-z_i} (-\frac{x_{ij}}{\sigma}) + (\alpha - 1) \sum_{i \in F} \left[\frac{m_i^2 e^{-2z_i} \gamma (-\frac{x_{ij}}{\sigma})}{[1-m_i]^2} + \frac{m_i e^{-z_i} (\frac{x_{ij}}{\sigma}) [1 - \gamma e^{-z_i}]}{[1-m_i]} \right] \\
&+ \alpha \sum_{i \in C} \left[\frac{m_i^2 e^{-2z_i} \gamma (-\frac{x_{ij}}{\sigma})}{[1-m_i]^2} + \frac{m_i e^{-z_i} (\frac{x_{ij}}{\sigma}) [1 - \gamma e^{-z_i}]}{[1-m_i]} \right];
\end{aligned}$$

$$\begin{aligned}
\mathrm{J}_{\sigma,\beta_j} = \frac{\partial^2 \mathrm{l}}{\partial \sigma \partial \beta_{\mathrm{j}}} &= -\sum_{i \in F} \frac{x_{ij}}{\sigma^2} + \gamma \sum_{i \in F} \left[-e^{-z_i} (\frac{z_i}{\sigma})(\frac{x_{ij}}{\sigma}) + e^{-z_i} (\frac{x_{ij}}{\sigma^2}) \right] \\
&+ (\alpha - 1) \sum_{i \in F} \left[\frac{\gamma^2 m_i^2 e^{-2z_i} (\frac{x_{ij}}{\sigma})(\frac{z_i}{\sigma})}{[1-m_i]^2} - \frac{\gamma m_i e^{-z_i} [\frac{x_{ij}}{\sigma^2} - (\frac{x_{ij}}{\sigma})(\frac{z_i}{\sigma}) + \gamma e^{-z_i}](\frac{x_{ij}}{\sigma})(\frac{z_i}{\sigma})}{[1-m_i]} \right] \\
&+ \alpha \sum_{i \in C} \left[\frac{\gamma^2 m_i^2 e^{-2z_i} (\frac{x_{ij}}{\sigma})(\frac{z_i}{\sigma})}{[1-m_i]^2} - \frac{\gamma m_i e^{-z_i} [\frac{x_{ij}}{\sigma^2} - (\frac{x_{ij}}{\sigma})(\frac{z_i}{\sigma}) + \gamma e^{-z_i}](\frac{x_{ij}}{\sigma})(\frac{z_i}{\sigma})}{[1-m_i]} \right];
\end{aligned}$$

For $j, s = 1, 2, ..., p$ and $j \neq s$

$$\mathrm{J}_{\beta_j,\beta_s} = \frac{\partial^2 \mathrm{l}}{\partial \beta_\mathrm{j} \partial \beta_\mathrm{s}} = -\gamma \sum_{i \in F} (\frac{x_{ij}}{\sigma})(\frac{x_{is}}{\sigma}) e^{-z_i}$$
$$+(\alpha - 1) \sum_{i \in F} \left[\frac{-\gamma^2 m_i^2 e^{-2z_i} (\frac{x_{ij}}{\sigma})(\frac{x_{is}}{\sigma})}{[1 - m_i]^2} - \frac{\gamma m_i e^{-z_i} (\frac{x_{ij}}{\sigma})(\frac{x_{ij}}{\sigma})[\gamma e^{-z_i} - 1]}{[1 - m_i]} \right]$$
$$+\alpha \sum_{i \in C} \left[\frac{-\gamma^2 m_i^2 e^{-2z_i} (\frac{x_{ij}}{\sigma})(\frac{x_{is}}{\sigma})}{[1 - m_i]^2} - \frac{\gamma m_i e^{-z_i} (\frac{x_{ij}}{\sigma})(\frac{x_{ij}}{\sigma})[\gamma e^{-z_i} - 1]}{[1 - m_i]} \right].$$

Case Weight Perturbation:

For $\widehat{z}_i = (\frac{y_i - \boldsymbol{x}_i{}^T \widehat{\boldsymbol{\beta}}}{\widehat{\sigma}})$, and $\widehat{m}_i = e^{\{-\widehat{\gamma} e^{-\widehat{z}_i}\}}$, the elements of $\boldsymbol{\Delta}$ matrix are as given below:

$$\Delta_{1i} = \frac{\partial^2 l(\boldsymbol{\theta}|w)}{\partial \alpha \partial w_i}\Big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}$$
$$= \left[\begin{array}{ll} \widehat{\alpha}^{-1} + \log[1 - \widehat{m}_i] & \text{if } i \in F, \\ \log[1 - \widehat{m}_i] & \text{if } i \in C. \end{array} \right.$$

$$\Delta_{2i} = \frac{\partial^2 l(\boldsymbol{\theta}|\boldsymbol{w})}{\partial \gamma \partial w_i}\Big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}$$
$$= \left[\begin{array}{ll} \widehat{\gamma}^{-1} - e^{-\widehat{z}_i} + \dfrac{(\widehat{\alpha} - 1)\widehat{m}_i e^{-\widehat{z}_i}}{[1 - \widehat{m}_i]} & \text{if } i \in F, \\ \dfrac{\widehat{\alpha}\widehat{m}_i e^{-\widehat{z}_i}}{[1 - \widehat{m}_i]} & \text{if } i \in C. \end{array} \right.$$

$$\Delta_{3i} = \frac{\partial^2 l(\boldsymbol{\theta}|\boldsymbol{w})}{\partial \sigma \partial w_i}\Big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}$$
$$= \left[\begin{array}{ll} \widehat{\sigma}^{-1}[-1 + \widehat{z}_i(1 - \widehat{\gamma} e^{-\widehat{z}_i})] + \dfrac{(\widehat{\alpha} - 1)\widehat{\gamma}\widehat{m}_i e^{-\widehat{z}_i}}{[1 - \widehat{m}_i]} & \text{if } i \in F, \\ \dfrac{\widehat{\alpha}\widehat{\gamma}\widehat{z}_i\widehat{m}_i e^{-\widehat{z}_i}}{[1 - \widehat{m}_i]} & \text{if } i \in C. \end{array} \right.$$

$$\Delta_{ji} = \frac{\partial^2 l(\boldsymbol{\theta}|\boldsymbol{w})}{\partial \beta_j \partial w_i}\Big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}, \; j = 4, ..., p + 3$$

Response Variable Perturbation:

For $\widehat{z_i}^* = \frac{(y_i + w_i S_y) - \boldsymbol{x}_i{}^T \widehat{\boldsymbol{\beta}}}{\widehat{\sigma}}$, and $\widehat{m_i}^* = e^{\{-\widehat{\gamma} e^{-\widehat{z_i}^*}\}}$, the elements of $\boldsymbol{\Delta}$ matrix can be written as:

$$\Delta_{1i} = \frac{\partial^2 l(\boldsymbol{\theta}|\boldsymbol{w})}{\partial \alpha \partial w_i}\Big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}$$
$$= \left[\begin{array}{ll} \dfrac{-\widehat{\sigma}^{-1}\widehat{\gamma} S_y e^{-\widehat{z_i}^*}\widehat{m_i}^*}{[1 - \widehat{m_i}^*]} & \text{if } i \in F, \\ \dfrac{-\widehat{\sigma}^{-1}\widehat{\gamma} S_y e^{-\widehat{z_i}^*}\widehat{m_i}^*}{[1 - \widehat{m_i}^*]} & \text{if } i \in C. \end{array} \right.$$

$$\Delta_{2i} = \frac{\partial^2 l(\boldsymbol{\theta}|\boldsymbol{w})}{\partial \gamma \partial w_i}\Big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}$$
$$= \left[\begin{array}{l} \widehat{\sigma}^{-1} S_y e^{-\widehat{z_i}^*} \left[1 + \dfrac{(\widehat{\alpha} - 1)\widehat{m_i}^*[\{\widehat{\gamma} e^{-\widehat{z_i}^*} - 1\}(1 - \widehat{m_i}^*) + \widehat{m_i}^* \widehat{\gamma} e^{-\widehat{z_i}^*}]}{[1 - \widehat{m_i}^*]^2} \right] \\ \hfill \text{if } i \in F, \\ \dfrac{\widehat{\alpha}\widehat{\sigma}^{-1} S_y e^{-\widehat{z_i}^*}\widehat{m_i}^*[\{\widehat{\gamma} e^{-\widehat{z_i}^*} - 1\}(1 - \widehat{m_i}^*) + \widehat{m_i}^* \widehat{\gamma} e^{-\widehat{z_i}^*}]}{[1 - \widehat{m_i}^*]^2} \\ \hfill \text{if } i \in C. \end{array} \right.$$

$$\Delta_{3i} = \frac{\partial^2 l(\boldsymbol{\theta}|\boldsymbol{w})}{\partial\sigma\partial w_i}\Big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}$$

$$= \begin{cases} \widehat{\sigma}^{-2}S_y[1+\widehat{\gamma}\widehat{z_i}^* e^{-\widehat{z_i}^*} - \widehat{\gamma}e^{-\widehat{z_i}^*}] \\ \quad + \dfrac{(\widehat{\alpha}-1)\widehat{\gamma}\widehat{\sigma}^{-2}S_y\widehat{m_i}^* e^{-\widehat{z_i}^*}[1-\widehat{z_i}^*+\widehat{\gamma}\widehat{z_i}^* e^{-\widehat{z_i}^*}]}{[1-\widehat{m_i}^*]} & \text{if } i \in F, \\ \quad + \dfrac{(\widehat{\alpha}-1)\widehat{\gamma}^2\widehat{\sigma}^{-2}S_y\widehat{z_i}^*(\widehat{m_i}^*)^2 e^{-2\widehat{z_i}^*}}{[1-\widehat{m_i}^*]^2} \\ \\ \dfrac{\widehat{\alpha}\widehat{\gamma}\widehat{\sigma}^{-2}S_y\widehat{m_i}^* e^{-\widehat{z_i}^*}[1-\widehat{z_i}^*+\widehat{\gamma}\widehat{z_i}^* e^{-\widehat{z_i}^*}]}{[1-\widehat{m_i}^*]} & \text{if } i \in C. \\ \quad + \dfrac{\widehat{\alpha}\widehat{\gamma}^2\widehat{\sigma}^{-2}S_y\widehat{z_i}^*(\widehat{m_i}^*)^2 e^{-2\widehat{z_i}^*}}{[1-\widehat{m_i}^*]^2} \end{cases}$$

$$\Delta_{ji} = \frac{\partial^2 l(\boldsymbol{\theta}|\boldsymbol{w})}{\partial\beta_j\partial w_i}\Big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}},\ j = 4, ..., p+3$$

$$= \begin{cases} x_{ij}\widehat{\sigma}^{-2}\widehat{\gamma}S_y e^{\widehat{z_i}^*} \\ \quad + \dfrac{(\widehat{\alpha}-1)\widehat{\sigma}^{-2}S_y x_{ij}\widehat{\gamma}\widehat{m_i}^* e^{-\widehat{z_i}^*}[\widehat{\gamma}e^{-\widehat{z_i}^*}-1]}{[1-\widehat{m_i}^*]} & \text{if } i \in F, \\ \quad + \dfrac{(\widehat{\alpha}-1)\widehat{\sigma}^{-2}S_y x_{ij}\widehat{\gamma}^2(\widehat{m_i}^*)^2 e^{-2\widehat{z_i}^*}}{[1-\widehat{m_i}^*]^2} \\ \\ \dfrac{\widehat{\alpha}\widehat{\sigma}^{-2}S_y x_{ij}\widehat{\gamma}\widehat{m_i}^* e^{-\widehat{z_i}^*}[\widehat{\gamma}e^{-\widehat{z_i}^*}-1]}{[1-\widehat{m_i}^*]} & \text{if } i \in C. \\ \quad + \dfrac{\widehat{\alpha}\widehat{\sigma}^{-2}S_y x_{ij}\widehat{\gamma}^2(\widehat{m_i}^*)^2 e^{-2\widehat{z_i}^*}}{[1-\widehat{m_i}^*]^2} \end{cases}$$

Explanatory Variable Perturbation: If $\widehat{z_i}^{**} = \dfrac{(y_i - \boldsymbol{x_i}^T\widehat{\boldsymbol{\beta}})}{\widehat{\sigma}}$, and $\widehat{m_i}^{**} = e^{\{-\widehat{\gamma}e^{-\widehat{z_i}^{**}}\}}$, then the elements of $\boldsymbol{\Delta}$ matrix can be written in the following forms:

$$\Delta_{1i} = \frac{\partial^2 l(\boldsymbol{\theta}|\boldsymbol{w})}{\partial\alpha\partial w_i}\Big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}$$

$$= \begin{cases} \dfrac{-\widehat{\sigma}^{-1}\widehat{\gamma}S_x\beta_t e^{-\widehat{z_i}^{**}}\widehat{m_i}^{**}}{[1-\widehat{m_i}^{**}]} & \text{if } i \in F, \\ \\ \dfrac{-\widehat{\sigma}^{-1}\widehat{\gamma}S_x\beta_t e^{-\widehat{z_i}^{**}}\widehat{m_i}^{**}}{[1-\widehat{m_i}^{**}]} & \text{if } i \in C. \end{cases}$$

$$\Delta_{2i} = \frac{\partial^2 l(\boldsymbol{\theta}|\boldsymbol{w})}{\partial\gamma\partial w_i}\Big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}$$

$$= \begin{cases} -\widehat{\sigma}^{-1}S_x\beta_t e^{-\widehat{z_i}^{**}}\left[1+\dfrac{(\widehat{\alpha}-1)\widehat{m_i}^{**}[\{\widehat{\gamma}e^{-\widehat{z_i}^{**}}-1\}(1-\widehat{m_i}^{**})+\widehat{m_i}^{**}\widehat{\gamma}e^{-\widehat{z_i}^{**}}]}{[1-\widehat{m_i}^{**}]^2}\right] \\ & \text{if } i \in F, \\ \\ \dfrac{-\widehat{\sigma}^{-1}S_x\beta_t e^{-\widehat{z_i}^{**}}\widehat{\alpha}\widehat{m_i}^{**}[\{\widehat{\gamma}e^{-\widehat{z_i}^{**}}-1\}(1-\widehat{m_i}^{**})+\widehat{m_i}^{**}\widehat{\gamma}e^{-\widehat{z_i}^{**}}]}{[1-\widehat{m_i}^{**}]^2} & \text{if } i \in C. \end{cases}$$

$$\Delta_{3i} = \frac{\partial^2 l(\boldsymbol{\theta}|\boldsymbol{w})}{\partial\sigma\partial w_i}\Big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}$$

$$= \begin{cases} \widehat{\sigma}^{-2}S_x[-1-\widehat{\gamma}\widehat{z}_i^{**}e^{-\widehat{z}_i^{**}}+\widehat{\gamma}e^{-\widehat{z}_i^{**}}] \\ \quad -\dfrac{(\widehat{\alpha}-1)\widehat{\gamma}\widehat{\sigma}^{-2}S_x\widehat{m_i}^{**}e^{-\widehat{z}_i^{**}}[1-\widehat{z}_i^{**}+\widehat{\gamma}\widehat{z}_i^{**}e^{-\widehat{z}_i^{**}}]}{[1-\widehat{m_i}^{**}]} & \text{if } i \in F, \\ \quad +\dfrac{(\widehat{\alpha}-1)\widehat{\gamma}^2\widehat{\sigma}^{-2}S_x\widehat{z}_i^{*}(\widehat{m_i}^{**})^2e^{-2\widehat{z}_i^{**}}}{[1-\widehat{m_i}^{**}]^2} \\ \\ \dfrac{-\widehat{\alpha}\widehat{\gamma}\widehat{\sigma}^{-2}S_x\widehat{m_i}^{**}e^{-\widehat{z}_i^{**}}[1-\widehat{z}_i^{**}+\widehat{\gamma}\widehat{z}_i^{**}e^{-\widehat{z}_i^{**}}]}{[1-\widehat{m_i}^{**}]} & \text{if } i \in C. \\ \quad +\dfrac{\widehat{\alpha}\widehat{\gamma}^2\widehat{\sigma}^{-2}S_x\widehat{z}_i^{**}(\widehat{m_i}^{**})^2e^{-2\widehat{z}_i^{**}}}{[1-\widehat{m_i}^{**}]^2} \end{cases}$$

$$\Delta_{ji} = \frac{\partial^2 l(\boldsymbol{\theta}|\boldsymbol{w})}{\partial\beta_j\partial w_i}\Big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}, \; j = 4, ..., p+3 \; and \; j \neq t,$$

$$= \begin{cases} -x_{ij}\widehat{\sigma}^{-2}\widehat{\gamma}S_x\widehat{\beta}_te^{-\widehat{z}_i^{**}} \\ \quad -\dfrac{(\widehat{\alpha}-1)\widehat{\gamma}\widehat{\sigma}^{-2}S_xx_{ij}\widehat{\beta}_t\widehat{m_i}^{**}e^{-\widehat{z}_i^{**}}[-1+\widehat{\gamma}e^{-\widehat{z}_i^{**}}]}{[1-\widehat{m_i}^{**}]} & \text{if } i \in F, \\ \quad -\dfrac{(\widehat{\alpha}-1)\widehat{\gamma}^2\widehat{\sigma}^{-2}S_xx_{ij}\widehat{\beta}_t(\widehat{m_i}^{**})^2\exp(-2\widehat{z}_i^{**})}{[1-\widehat{m_i}^{**}]^2} \\ \\ -\dfrac{\widehat{\alpha}\widehat{\gamma}\widehat{\sigma}^{-2}S_xx_{ij}\widehat{\beta}_t\widehat{m_i}^{**}e^{-\widehat{z}_i^{**}}[-1+\widehat{\gamma}e^{-\widehat{z}_i^{**}}]}{[1-\widehat{m_i}^{**}]} & \text{if } i \in C. \\ \quad -\dfrac{\widehat{\alpha}\widehat{\gamma}^2\widehat{\sigma}^{-2}S_xx_{ij}\widehat{\beta}_t(\widehat{m_i}^{**})^2e^{-2\widehat{z}_i^{**}}}{[1-\widehat{m_i}^{**}]^2} \end{cases}$$

and

$$\Delta_{ti} = \frac{\partial^2 l(\boldsymbol{\theta}|\boldsymbol{w})}{\partial\beta_t\partial w_i}\Big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}$$

$$= \begin{cases} S_x\widehat{\sigma}^{-1}[1-\widehat{\gamma}\widehat{\sigma}^{-1}\widehat{\beta}_t(x_{it}+w_iS_x)e^{-\widehat{z}_i^{**}}-\widehat{\gamma}e^{-\widehat{z}_i^{**}}] \\ \quad +\dfrac{(\widehat{\alpha}-1)\widehat{\gamma}\widehat{\sigma}^{-1}S_x\widehat{m_i}^{**}e^{-\widehat{z}_i^{**}}[1+\widehat{\beta}_t(\frac{x_{it}+w_iS_x}{\sigma})\{1-\widehat{\gamma}e^{-\widehat{z}_i^{**}}\}]}{[1-\widehat{m_i}^{**}]} & \text{if } i \in F, \\ \quad -\dfrac{(\widehat{\alpha}-1)\widehat{\gamma}^2\widehat{\sigma}^{-2}\widehat{\beta}_tS_x(\widehat{m_i}^{**})^2e^{-2\widehat{z}_i^{**}}(x_{it}+w_iS_x)}{[1-\widehat{m_i}^{**}]^2} \\ \\ \dfrac{\widehat{\alpha}\widehat{\gamma}\widehat{\sigma}^{-1}S_x\widehat{m_i}^{**}e^{-\widehat{z}_i^{**}}[1+\widehat{\beta}_t(\frac{x_{it}+w_iS_x}{\sigma})\{1-\widehat{\gamma}e^{-\widehat{z}_i^{**}}\}]}{[1-\widehat{m_i}^{**}]} & \text{if } i \in C. \\ \quad -\dfrac{\widehat{\alpha}\widehat{\gamma}^2\widehat{\sigma}^{-2}\widehat{\beta}_tS_x(\widehat{m_i}^{**})^2e^{-2\widehat{z}_i^{**}}(x_{it}+w_iS_x)}{[1-\widehat{m_i}^{**}]^2} \end{cases}$$

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