SCOPUA Journal of Applied Statistical Research (JASR)

Vol.2, Issue 2, June 2026
DOI: https://doi.org/10.64060/JASR2V2i5

Development and Statistical Analysis of Modified Kies Perks Distribution

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Received: 18 March 2026 / Revised: 01 May 2026 / Accepted: 10 May 2026 / Published online: 17 May 2026

ABSTRACT

This paper introduces a novel development of the Perks Distribution called the Modified Kies Perks distribution (MKPD), a highly adaptable probabilistic model characterized by its versatile probability density function, distribution function (DF) and hazard rate function. The MKPD can take various shapes, such as a single peak (unimodal), leaning to the right (right skewed), leaning to the left (left-skewed), a curve resembling a bathtub, and more, demonstrating its flexibility in statistical modelling and it is one of the most important and well-established ideas in reliability analysis. Several fundamental characteristics of the proposed model have been derived, including its PDF, CDF, quantile function, Hazard rate, survival function, cumulative hazard function, Odds Ratio and Average Failure Rate. Furthermore, a Monte Carlo simulation study is performed to assess the efficiency of different estimation methods across varying sample sizes confirming the MKPD's flexibility and strong fit for lifetime, epidemiological and actuarial data. To validate its practicality, the proposed distribution is applied to real datasets. The results demonstrate that the MKPD is a valuable contribution to both statistical theory and applied statistical work.

Keywords*:*Perk distribution; Modified Kies distribution; estimation; Reliability measures

1. Introduction

Statistical modelling of lifetime and reliability data plays an essential role in many scientific disciplines, including engineering, biomedical sciences, actuarial studies, and risk analysis. Reliable statistical models enable researchers to analyse failure mechanisms, estimate survival probabilities, and make informed decisions regarding system reliability. Classical probability distributions, such as exponential, gamma, and Weibull distributions, have been widely used in reliability and survival analysis due to their mathematical simplicity and interpretability. However, these classical models often lack sufficient flexibility to represent complex data patterns, particularly when the hazard rate function exhibits non-monotonic behaviours such as bathtub-shaped or unimodal patterns (Kotz & Balakrishnan, 2012).

The development of flexible lifetime distributions has been a central focus in reliability and actuarial analysis. Early contributions by William Perks laid the foundation through the introduction of the Perks distribution for modeling mortality data (Perks, 1932), while Harold A. Kies extended this framework by proposing the Kies family of distributions for life testing applications (Kies, 1958, 1959). Subsequent studies have introduced several modifications and generalizations of the Kies distribution to enhance its flexibility and applicability to lifetime data, including modified and power versions that accommodate diverse data behaviors (El-Bassiouny et al., 2016; Yousof et al., 2017; Shanker & Mishra, 2013). In parallel, a wide range of classical distributions such as the Weibull, exponential, beta, and Pareto distributions have been extensively studied and applied in modeling failure rates and survival data (Weibull, 1951; Balakrishnan & Basu, 1995; Nadarajah & Kotz, 2003; Arnold, 2015).

More recently, researchers have focused on developing generalized and transmuted families of distributions using innovative generator techniques to improve modeling flexibility and capture complex characteristics such as skewness, heavy tails, and varying hazard rate shapes (Cordeiro & de Castro, 2011; Nofal et al., 2017; Afify et al., 2016). Methods for generating new distribution families have also been widely explored, enabling the construction of models with enhanced adaptability for real-world data (Alzaatreh et al., 2013; Baharith & Aljuhani, 2021). Contemporary studies continue to advance this area by proposing new lifetime distributions through transformation and stochastic generator approaches, demonstrating improved performance in biomedical, environmental, and engineering applications (Bakr et al., 2022; Elbatal et al., 2022; Yusuf et al., 2024; Dutta & Yadav, 2025). Despite these advancements, there remains a need for more flexible models that combine rich shape characteristics with tractable properties and robust estimation performance. This gap motivates further extensions of the Kies–Perks framework, leading to the development of more versatile models such as the Modified Kies Perks Distribution.

The Weibull distribution, introduced by Waloddi Weibull, has been extensively applied for modelling failure time data due to its flexibility in representing increasing and decreasing hazard rates. Similarly, the Lindley distribution, proposed by Dennis V. Lindley, has found widespread applications in reliability analysis and Bayesian statistics. Despite their usefulness, these classical distributions are sometimes inadequate for modelling real datasets with complex shapes, skewness, and varying failure patterns. One of the earliest contributions to actuarial and mortality modelling was made by William Perks, who introduced the Perks distribution to model mortality rates in life tables. The Perks model improved representation of mortality patterns compared with simpler models used at the time. Later, Harold A. Kies introduced the Kies distribution for life testing experiments, providing a flexible framework for constructing probability models capable of capturing diverse failure behaviors. Subsequent research explored extensions of the Kies distribution to improve flexibility and applicability. For instance, Abd El-Basit El-Bassiouny and colleagues proposed a modified Kies distribution with additional parameters to capture complex lifetime patterns, while Hesham M. Yousof and co-authors developed the power Kies distribution through power transformations. In parallel, significant research has focused on developing generalized families of distributions by applying transformation or generator mechanisms to baseline models. Generator-based approaches introduce additional shape parameters that enhance modelling flexibility. The method proposed by Fathi Alzaatreh et al. (2013) has been widely adopted to construct new families of probability distributions, and generalized distribution families developed by Gauss M. Cordeiro and collaborators have attracted attention for modeling skewed and heavy-tailed data. Recent studies, including those by Elbatal et al. (2022), Bakr et al. (2022), Baharith and Aljuhani (2021), Yusuf et al. (2024), Dutta and Yadav (2025), Mecha et al. (2025), and Pandey et al. (2024), have demonstrated the effectiveness of generator-based methods in constructing flexible distributions for engineering, biomedical, and actuarial applications.

Despite these advances, there remains a need for distributions that combine the structural characteristics of classical models with the flexibility of modern generator techniques. Motivated by these developments, this study proposes a new probability distribution called the Modified Kies Perks Distribution (MKPD). The MKPD integrates the structural properties of the Perks distribution with the flexibility of the Kies transformation, introducing additional parameters that allow the probability density function and hazard rate function to assume various shapes. This makes the MKPD suitable for modelling diverse types of lifetime data. The main objectives of this study are to develop the mathematical formulation of the MKPD, investigate its statistical properties, and perform parameter estimation using the maximum likelihood method. A Monte Carlo simulation study is conducted to evaluate estimator performance, and the applicability of the proposed model is demonstrated using real datasets. The proposed approach provides a flexible and robust framework for lifetime and reliability analysis, addressing limitations of both classical and existing generalized distributions.

The growing complexity of real-world data in fields such as reliability analysis, survival studies, epidemiology, and actuarial science has created a strong demand for flexible probability distributions capable of capturing diverse behaviors including skewness, heavy tails, and varying hazard rate shapes such as increasing, decreasing, and bathtub forms. Although classical models, including the Perks distribution, have been widely used due to their theoretical importance, they often lack the flexibility required to accurately model such complex data patterns. Existing extensions of these models also remain limited, as many fail to simultaneously provide adaptable shape characteristics, tractable mathematical properties, and strong performance across different datasets. Furthermore, there is a noticeable lack of comprehensive studies that evaluate multiple estimation techniques under varying sample sizes using simulation methods, along with limited real data validation to confirm practical applicability. These limitations highlight a clear research gap in developing a more versatile and efficient model. To address this gap, the Modified Kies Perks Distribution (MKPD) is proposed as a highly flexible extension that accommodates a wide range of distributional shapes and hazard rate behaviors, while also offering well-defined statistical properties and improved estimation performance, making it suitable for both theoretical development and practical applications

2. Methodology

Unser this section the development of the MKPD is explained, the PDF and CDF of the proposed models along with graphical representation is shown.

2.1 The Modified Kies (MK) Family

The Modified Kies family is a modern class of statistical distributions designed to extend classical lifetime models, offering greater flexibility in reliability analysis, survival studies, and actuarial applications. It generalizes baseline distributions to capture diverse hazard rate shapes, including monotonic, non-monotonic, and bathtub curves. Al-Babtain et al (2020) proposed a modified Kies (MKi) family with the CDF and PDF as follows:

$$\mathrm{F}(x;\omega) = 1 - \exp\left(-\left[\frac{\mathrm{G}(x;\omega)}{1-\mathrm{G}(x;\omega)}\right]^{\gamma}\right); \mathrm{x} > 0, \gamma > 0 \tag{1}$$

$$f(x;\omega) = \gamma \cdot \mathrm{g}(x;\omega) \cdot [\mathrm{G}(x;\omega)]^{\gamma-1}. \quad [1 - \mathrm{G}(x;\omega)]^{-(\gamma+1)} \cdot \exp\left(-\left[\frac{\mathrm{G}(x;\omega)}{1-\mathrm{G}(x;\omega)}\right]^{\gamma}\right); \quad x > 0,\ \gamma > 0 \tag{2}$$

2.2 Background of the Perks Distribución

The Perks distribution is a flexible statistical model widely applied in reliability engineering and actuarial science, particularly for analyzing lifetime data. Its distinctive feature lies in its ability to capture both monotonic and non-monotonic failure rate behaviors, including the well-known bathtub-shaped hazard function. In this study, we adopt the Perks distribution, parameterized by α and λ, as the baseline model for subsequent analysis.

So, the CDF and PDF of Perks Distribution are

$$G(x) = \frac{\alpha(e^{\lambda x}-1)}{1+\alpha e^{\lambda x}} \ ; \quad \alpha, \lambda > 0, x > 0 \tag{3}$$

$$g(x) = \frac{\alpha\lambda e^{\lambda x}(1+\alpha)}{(1+\alpha e^{\lambda x})^{2}}; \quad \alpha, \lambda > 0, x > 0 \tag{4}$$

Where α, λ *are shape and scale parameter respectively*

2.3 Formulation of the Proposed Distribution

By embedding the Perks distribution within the Modified Kies family framework, this study establishes a robust baseline for comparative analysis of lifetime models. This integration enables a systematic exploration of their structural properties, parameter estimation methods, and practical applications in reliability and survival contexts. Specifically, we propose the Modified Kies Perks Distribution (MKPD), constructed by applying the Kies transformation to the Perks distribution. The resulting model is characterized by three positive parameters, α, λ, and γ, which collectively enhance its flexibility in capturing diverse hazard rate behaviors

Using the CDF and PDF in equations (3) and (4) in equations (1) and (2), the cumulative distribution function (CDF) and probability density function (PDF)of the MKPD is defined as

$$F(x;\alpha,\lambda,\gamma) = 1 - exp\left(-\left(\frac{\alpha(e^{\lambda x}-1)}{1+\alpha}\right)^{\gamma}\right); x > 0, \alpha, \lambda, \gamma > 0 \tag{4}$$

$$f(x;\alpha,\lambda,\gamma) = \frac{\gamma\alpha^{\gamma}\lambda e^{\lambda x}}{(1+\alpha)^{\gamma}}.\left(e^{\lambda x}-1\right)^{\gamma-1} exp\left(-\left(\frac{\alpha(e^{\lambda x}-1)}{1+\alpha}\right)^{\gamma}\right); x > 0, \alpha, \lambda, \gamma > 0 \tag{5}$$

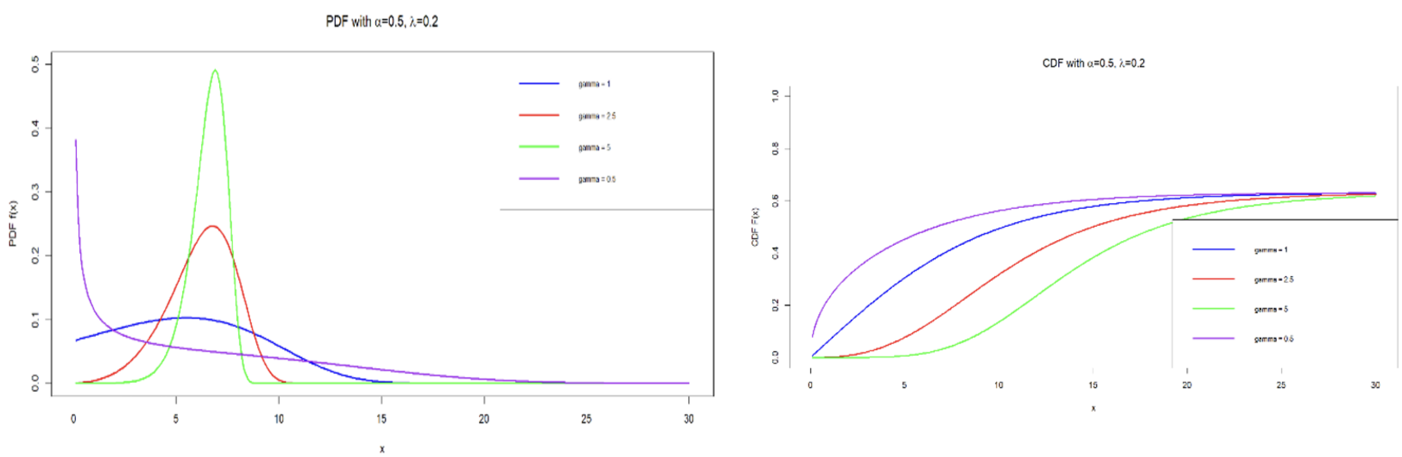


Figure 1: PDF plot of MKPD (left) and CDF plot of MKPD (right)

Theorem 1. Show that the CDF in Eq. (4) is valid CDF.

Proof. Consider X follows the MKPD with CDF in Eq. (4), then

$$F(-\infty) = F(0) = 1 - exp\left(-\left(\frac{\alpha\left(e^{\lambda(0)}-1\right)}{1+\alpha}\right)^{\gamma}\right) = 1 - Exp(0) = 0$$

$$F(\infty) = 1 - exp\left(-\left(\frac{\alpha\left(e^{\lambda(\infty)}-1\right)}{1+\alpha}\right)^{\gamma}\right) = 1 - Exp(-\infty) = 1$$

So, CDF in Eq. (4) is a valid CDF.

3. Properties of MKPD

In this section, some properties of the MKPD are presented.

Hazard Rate Function

$$h(x) = \frac{\gamma \alpha^{\gamma} \lambda e^{\lambda x}}{(1+\alpha)^{\gamma}} . \left(e^{\lambda x} - 1\right)^{\gamma - 1} \tag{6}$$

- for $\gamma = 1, :\ h(x) = \frac{\alpha \lambda e^{\lambda x}}{(1+\alpha)}$;
- for $\gamma = 1\ and\ \alpha = 1; h(x) = \frac{\lambda e^{\lambda x}}{2}$

Survival Function

$$S(x) = exp\left(-\left(\frac{\alpha(e^{\lambda x} - 1)}{1+\alpha}\right)^{\gamma}\right) \tag{7}$$

Cumulative Hazard Function

$$H(x) = \left(\frac{\alpha(e^{\lambda x} - 1)}{1+\alpha}\right)^{\gamma} \tag{8}$$

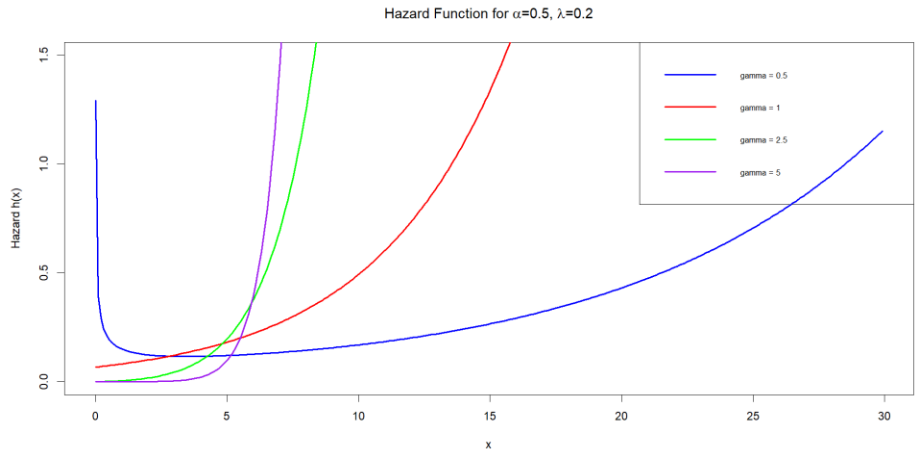


Figure 2: HRF plot of MKPD

Odds Ratio Function

$$O(x) = exp\left[\left(\frac{\alpha(e^{\lambda x} - 1)}{1+\alpha}\right)^{\gamma}\right] - 1 \tag{9}$$

Average Failure Rate

$$ARF(x) = \frac{\left(\frac{\alpha(e^{\lambda x} - 1)}{1+\alpha}\right)^{\gamma}}{x} \tag{10}$$

This equation has not closed-form solution, so the mode is obtained numerically by maximizing f(x)

Quantile Function

$$Q(p) = \frac{1}{\lambda} \ln\left(\frac{1+\alpha}{\alpha}(-\ln(1-p))^{\frac{1}{\gamma}} + 1\right); 0 < p < 1 \tag{11}$$

The median, upper and lower quartiles are obtained, respectively, by substituting 0.5,0.25 and 0.75 into the quantile function. The Bowley's measure of skewness and Moors's measure of kurtosis can be calculated using the quantiles. They are given by

$$B_{SK} = \frac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1}$$

and

$$M_K = \frac{Q(0.375; \alpha, \lambda, \gamma) - Q(0.125; \alpha, \lambda, \gamma) + Q(0.875; \alpha, \lambda, \gamma) - Q(0.625; \alpha, \lambda, \gamma)}{Q(0.75; \alpha, \lambda, \gamma) - Q(0.25; \alpha, \lambda, \gamma)}$$

The MKPD is a highly flexible distribution, the three parameters $(\alpha, \lambda, \gamma)$ jointly control the skewness direction and tail heaviness independently, making it capable of fitting a very wide variety of real-world lifetime and survival datasets without being locked into one shape family.

The plots of the Bowley's coefficient of skewness and Moor's coefficient of kurtosis are displayed in figure 3. Both the skewness and kurtosis are affected by the change in the values of the parameters. From the figure, we can observe that the MKPD can be left or right skewed.

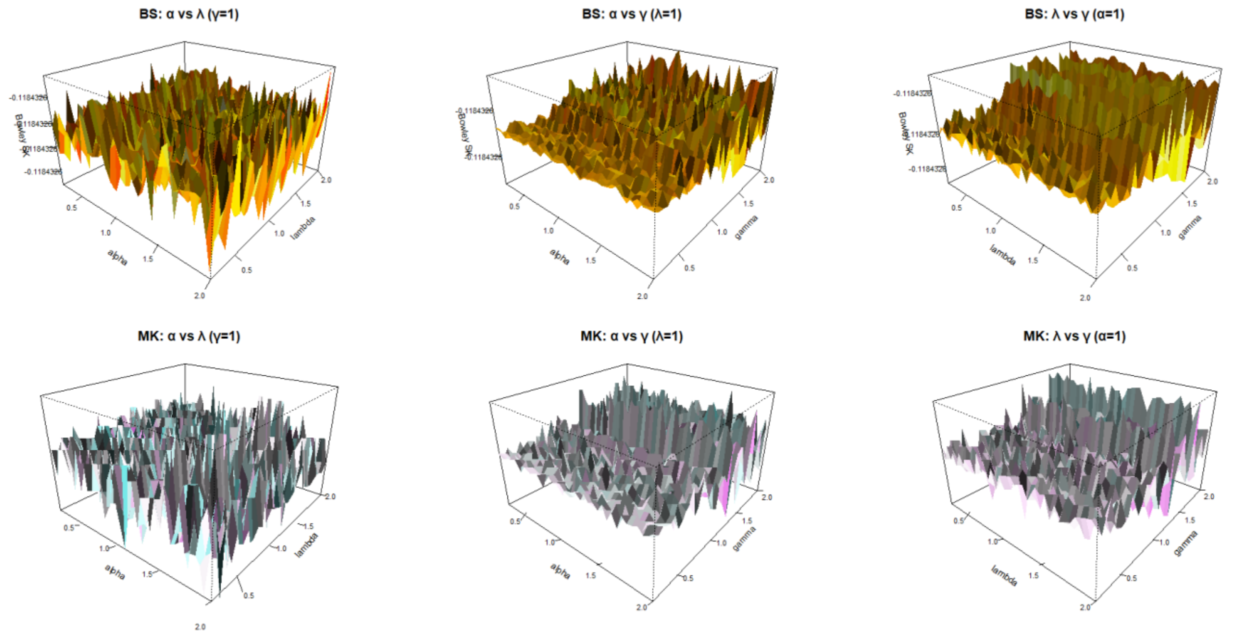


Figure 3: Bowley's skewness 3-D plots(above) and Moor's kurtosis 3-D plots(below)

The r[th] raw moment of a random variable $X \sim MKPD(\alpha, \lambda, \gamma)$is:

$$\mu_r' = E(X^r) = \frac{\gamma\alpha^\gamma\lambda}{(1+\alpha)^\gamma}\int_0^\infty x^r e^{\lambda x}\left(e^{\lambda x}-1\right)^{\gamma-1} exp\left(-\left(\frac{\alpha\left(e^{\lambda x}-1\right)}{1+\alpha}\right)^\gamma\right)dx$$

$$= \frac{\gamma\alpha^\gamma\lambda}{(1+\alpha)^\gamma}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!}\sum_{k=0}^{\gamma-1}\binom{\gamma-1}{k}(-1)^{\gamma-k-1}\int_0^\infty x^{r+i}\left(e^{\lambda x}\right)^k xp\left(-\left(\frac{\alpha(e^{\lambda x}-1)}{1+\alpha}\right)^\gamma\right)dx \qquad (12)$$

Since closed form of the expression in Eq. (12) for the moments of the proposed distribution are not tractable, they are evaluated numerically using appropriate integration techniques. This approach ensures reliable approximation of the distribution's descriptive statistics and higher-order moments for practical applications.

Table1: Moments and Descriptive Statistics of the Modified Kies Perks Distribution (MKPD)

α	λ	γ	Mean	SD	E(X)	E(X^2)	E(X^3)	E(X^4)	Skewness	Kurtosis
0.5	1.0	1.0	1.1568	0.6659	1.156806	1.781624	3.168428	6.190837	0.2761	2.3521
1.0	1.0	2.0	0.9636	0.3391	0.963561	1.043413	1.220332	1.510993	-0.1695	2.5738
0.8	1.5	1.0	0.6584	0.4006	0.658446	0.594037	0.627401	0.734213	0.3876	2.4518
1.2	0.8	2.0	1.1407	0.4086	1.140724	1.468189	2.046006	3.023858	-0.1415	2.5578
1.5	1.2	1.5	0.7005	0.3266	0.700479	0.597316	0.572393	0.595249	0.1316	2.4346

Table 1 shows the Descriptive statistics of the MKPD for selected parameter values. The distribution demonstrates considerable flexibility, as its Mean and variance vary with changes in the parameters, allowing it to effectively capture differences in both location and dispersion. The presence of both positive and negative skewness values indicates its ability to model data with right-skewed as well as left-skewed characteristics. Furthermore, the kurtosis values, ranging between approximately 2.35 and 2.57, suggest a moderately light-tailed behavior compared to the normal distribution, highlighting its suitability for applications where lighter tails are desired. Also, the results show that the magnitude of the moments varies with the parameters, indicating the flexibility of the proposed distribution in modelling different data behaviors.

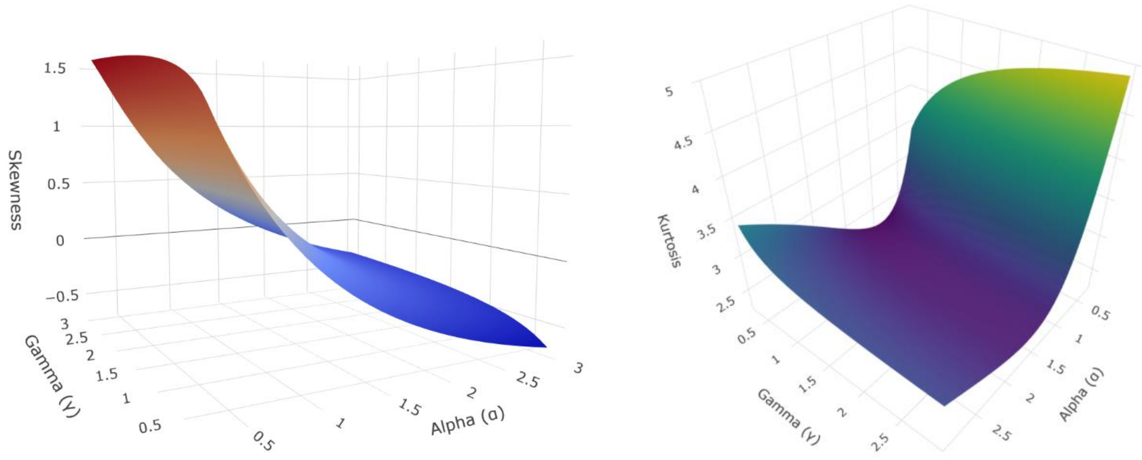


Figure 4: skewness(left) and kurtosis (right) plots

The moment generating function of a random variable X is

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x)\, dx$$

The moment generating function of the Modified Kies Perks Distribution does not admit a simple closed form. However, using the transformation can be expressed as an infinite series involving the Gamma function.

The Shannon entropy of a continuous distribution is

$$H(x) - E[\ln f(x)] = -\int_0^\infty f(x) \ln f(x)\, dx$$

For the MKPD, using the pdf the entropy becomes

$$H(x) = -\int_0^\infty f(x) \ln\left[\frac{\gamma\alpha^\gamma \lambda e^{\lambda x}}{(1+\alpha)^\gamma} . \left(e^{\lambda x} - 1\right)^{\gamma-1} . exp\left(-\left(\frac{\alpha(e^{\lambda x}-1)}{1+\alpha}\right)^\gamma\right)\right] dx \qquad (13)$$

The expression of Shannon entropy in Eq. (13) is not in close form and therefore is usually evaluated numerically. Table 2 illustrates that overall, the variation in Shannon entropy demonstrates the flexibility of the MKPD in modelling data with different levels of uncertainty.

Table 2: Shannon entropy measure of the (MKPD) for different parameter values

α	λ	γ	Shannon Entropy
0.5	1.0	1.0	0.9418
1.0	1.0	2.0	0.3250
0.8	1.5	1.0	0.4178
1.2	0.8	2.0	0.5122
1.5	1.2	1.5	0.2749

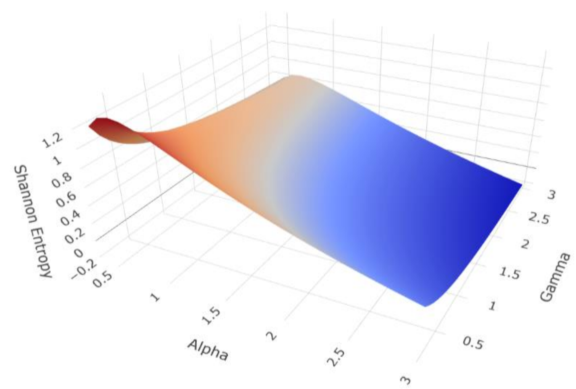


Figure 5: 3D plot of Shannon Entropy

4. Estimation Methods and a Simulation Study

In this section the estimation procedure by different methods of estimations of the proposed distribution is presented.

4.1 Maximum Likelihood Estimation (MLE)

Consider the random variables $X_1, \dots, X_n$ follows the MKPD in Eq.(5), then the log likelihood function of it is:

$$\ell(\alpha,\lambda,\gamma) = \log L = \sum_{i=1}^{n} \log f(X_i;\alpha,\lambda,\gamma)$$

$$\log L = nlog(\gamma) + nlog(\lambda) + n\gamma log(\alpha) - n\gamma log(1+\alpha) + \lambda x - \left(\frac{\alpha(e^{\lambda x}-1)}{1+\alpha}\right)^{\gamma} \tag{14}$$

$$\frac{\partial \ell}{\partial \alpha} = 0, \frac{\partial \ell}{\partial \lambda} = 0, \frac{\partial \ell}{\partial \gamma} = 0$$

The score functions do not have closed forms; thus, the equation is solved numerically to obtain the estimates. The MLE can be obtained by solving the log likelihood function by taking the derivative with respect to the parameters of MKPD but due to complexity of the expression the derivative expression is not in close form, and the parameters estimation is done by using the R language:

4.2 Ordinary and weighted least squares estimators

Ordinary Least Squares Estimation (OLSE) and Weighted Least Squares Estimation (WLSE) represent two principal techniques for parameter estimation in linear models. OLSE is grounded in the assumption of homoscedasticity, seeking to minimize the sum of squared deviations between observed and predicted values, thereby yielding efficient estimates when error variances are constant across observations. In contrast, WLSE is designed to address heteroscedasticity, where variances differ among data points. By assigning weights proportional to the reliability of each observation, WLSE ensures that more precise data exert greater influence on the estimation process. This weighting mechanism enhances the robustness and accuracy of parameter estimates, making WLSE particularly suitable in contexts where unequal variances would otherwise compromise model validity.

The ordered sample $x_{(1),}x_{(2)}, \dots \dots \dots, x_{(n)}$ is obtained from a basic random sample of size n. Minimizing the following function with respect to the MKPD parameters yields the OLSE estimation of the parameter as follows:

$$OLS_{(\alpha,\lambda,\gamma)} = \sum_{i=1}^{n} \left[F(x) - \frac{i}{n+1}\right]^2$$

$$\Rightarrow OLS_{(\alpha,\lambda,\gamma)} = \sum_{i=1}^{n} \left[1 - exp\left(-\left(\frac{\alpha\left(e^{\lambda x}-1\right)}{1+\alpha}\right)^{\gamma}\right) - \frac{i}{n+1}\right]^2$$

Minimizing the following function with respect to the MKPD parameters yield the WLSE estimations of the parameter as follow

$$WLS_{(\alpha,\lambda,\gamma)} = \sum_{i=1}^{n} w_i \left[F(x) - \frac{i}{n+1}\right]^2$$

Here

$$w_i = \frac{(n+1)^2(n+2)}{i(n-i+1)}$$

$$WLS_{(\alpha,\lambda,\gamma)} = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[1 - exp\left(-\left(\frac{\alpha\left(e^{\lambda x}-1\right)}{1+\alpha}\right)^{\gamma}\right) - \frac{i}{n+1}\right]^2$$

4.3 Cramér-von Mises Estimation (CVM)

The Cramér–von Mises Estimator (CVME) seeks to minimize the difference between a model's theoretical cumulative distribution function (CDF) and the empirical distribution function (EDF) of the sample. The CVME for the MKPD parameters (α,λ,γ) is obtained by minimizing the following function.

$$CVM_{(\alpha,\lambda,\gamma)} = \frac{1}{12n} + \sum_{i=1}^{n} \left[F(x) - \frac{2i-1}{2n}\right]^2$$

$$CVM_{(\alpha,\lambda,\gamma)} = \frac{1}{12n} + \sum_{i=1}^{n} \left[1 - exp\left(-\left(\frac{\alpha\left(e^{\lambda x}-1\right)}{1+\alpha}\right)^{\gamma}\right) - \frac{2i-1}{2n}\right]^2$$

4.4 Anderson Darling estimators

The Anderson-Darling Estimator (ADE) minimizes the difference between the empirical distribution function (EDF) and the theoretical cumulative distribution function (CDF), with stronger emphasis on the distribution's tails. For a sample of size (n), denoted as $x_{(1),}x_{(2)}, \dots\dots\dots, x_{(n)}$ the ADE of the MKPD parameters $(\alpha, \lambda, \gamma)$ is obtained by minimizing the following function.

$$AD_{(\alpha,\lambda,\gamma)} = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln F(x) + \ln\left(1 - F(x)\right)\right]$$

$$AD_{(\alpha,\lambda,\gamma)} = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln\left(1 - exp\left(-\left(\frac{\alpha\left(e^{\lambda x}-1\right)}{1+\alpha}\right)^{\gamma}\right)\right) + \ln\left(exp\left(-\left(\frac{\alpha\left(e^{\lambda x}-1\right)}{1+\alpha}\right)^{\gamma}\right)\right)\right]$$

In R, the optimization function can be employed to numerically compute parameter estimates $(\alpha, \lambda, \gamma)$ for all estimation methods based on the MKPD model.

5. Simulation Study

This section employs simulated data to assess the efficacy of the proposed estimate methods for ascertaining the parameters of the MKPD model by generating 1000 iterations from the MKPD model. Simulations are conducted across varying sample sizes $(n = 25,65,125,200)$ and parameters configurations. The parameter sets for $(\alpha, \lambda, \gamma)$ included $(0.6, 0.5, 0.6)$, $(0.6, 0.5, 3)$, $(0.6, 2.6, 0.6)$, $(0.6, 2.6, 2.4)$, $(2.8, 0.8, 0.6)$, $(2.8, 0.8, 2.4)$, $(2.8, 2.7, 0.4)$, and $(2.8, 2.7, 3)$.

The tables compare five estimation methods. They are evaluated using:

- RMSE (Root Mean Square Error) → measures accuracy
- RAB (Relative Absolute Bias) → measures bias

Smaller values of RMSE and RAB indicate better performance.

Algorithm for Monte Carlo Simulation of MKPD

Step 1: Specify true parameter values

Choose the true values of parameters:

- α, λ, γ

  For example:
- (0.6, 0.5, 0.6), (0.6, 0.5, 3), etc.

Step 2: Select sample sizes

Define different sample sizes:

- $n = 25,65,125,200$

Step 3: Fix number of simulations

Set number of Monte Carlo replications:

- $N = 1000$(or any large number)

Step 4: Generate random samples

For each simulation $i = 1,2,\dots,N$:

1. Generate a random sample of size $n$from MKPD using:
   - inverse transformation method **or**
   - any numerical sampling technique suitable for MKPD
2. Denote the sample as:

$$X_1, X_2, \dots, X_n$$

Step 5: Estimate parameters

For each generated sample, compute estimates using:

- MLE (Maximum Likelihood Estimation)
- OLS (Ordinary Least Squares)
- WLS (Weighted Least Squares)
- CVM (Cramér–von Mises)
- AD (Anderson–Darling)

Obtain:

- $\hat{\alpha}, \hat{\lambda}, \hat{\gamma}$

Step 6: Repeat simulation

Repeat Steps 4 and 5 for all Nsimulations and store estimates.

Step 7: Compute performance measures

For each estimator and parameter:
Root Mean Square Error (RMSE)

$$RMSE(\theta) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2}$$

Relative Absolute Bias (RAB)

$$RAB(\theta) = \frac{1}{N}\sum_{i=1}^{N} \left| \frac{\hat{\theta}_i - \theta}{\theta} \right|$$

where $\theta \in \{\alpha, \lambda, \gamma\}$
Relative absolute bias (RAB) and root mean square error (RMSE) were computed using R and summarized in Tables 3,4,5,6,7,8,9 and 10 to assess the performance of different estimation methods. This simulation study, based on multiple iterations, evaluates estimator accuracy across varying sample sizes and identifies the most effective approach for parameter estimation. Results show consistent reductions in RAB and RMSE with larger samples, confirming the reliability and robustness of the MKPD model. The findings highlight its stability and practical relevance for analyzing complex lifetime data with nonmonotonic hazard rates, particularly in fields such as reliability engineering, healthcare, and survival analysis. The Monte Carlo simulation results for the Modified Kies Perks Distribution (MKPD) clearly show that the performance of all estimation methods improve as the sample size increases, since both RMSE and RAB consistently decrease when moving from smaller samples to larger ones, indicating consistency of the estimators . Among the methods, Maximum Likelihood Estimation (MLE) generally provides the most accurate and reliable estimates, exhibiting the smallest RMSE and bias in most parameter settings, while Weighted Least Squares (WLS) often performs as the second-best method and remains close to MLE in efficiency. Ordinary Least Squares (OLS) shows moderate performance but is usually less accurate than MLE and WLS, whereas the Cramér–von Mises (CVM) and Anderson–Darling (AD) methods tend to be less stable and produce comparatively higher errors, although AD occasionally performs better for estimating the parameter γ. The results also indicate that estimation becomes more difficult when the parameter values, particularly λ and γ, are large, as this leads to higher RMSE and RAB values across all methods. Furthermore, the parameter α appears to be the most challenging to estimate accurately, while γ is often estimated with relatively better precision in some cases. Overall, the study demonstrates that MLE is the most efficient estimation technique for MKPD, especially with larger sample sizes, while WLS serves as a strong alternative, and the accuracy of all methods depends significantly on both the sample size and the underlying parameter values.

Table 3: Estimation methods with different sample sizes where $(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\gamma}) = (\mathbf{0.6}, \mathbf{0.5}, \mathbf{0.6})$

Estimator	α	λ	γ	n	RMSE α	RMSE λ	RMSE γ	RAB α	RAB λ	RAB γ
MLE	0.6	0.5	0.6	25	4.734	0.315	0.187	4.937	0.425	0.257
OLS	0.6	0.5	0.6	25	5.491	0.424	0.194	6.362	0.559	0.270
WLS	0.6	0.5	0.6	25	5.329	0.373	0.178	5.931	0.497	0.251
CVM	0.6	0.5	0.6	25	5.167	0.331	0.179	5.023	0.435	0.250
AD	0.6	0.5	0.6	25	5.038	0.383	0.127	4.950	0.552	0.210
MLE	0.6	0.5	0.6	65	3.635	0.192	0.130	3.229	0.272	0.183
OLS	0.6	0.5	0.6	65	4.570	0.271	0.142	4.668	0.386	0.206
WLS	0.6	0.5	0.6	65	4.204	0.226	0.127	4.031	0.321	0.180
CVM	0.6	0.5	0.6	65	3.735	0.292	0.120	4.229	0.372	0.213
AD	0.6	0.5	0.6	65	4.038	0.323	0.110	4.450	0.452	0.200
MLE	0.6	0.5	0.6	125	2.920	0.154	0.107	2.320	0.220	0.153
OLS	0.6	0.5	0.6	125	3.574	0.205	0.116	3.223	0.293	0.165
WLS	0.6	0.5	0.6	125	3.152	0.171	0.105	2.632	0.246	0.150
CVM	0.6	0.5	0.6	125	3.635	0.252	0.110	4.130	0.321	0.202
AD	0.6	0.5	0.6	125	3.738	0.305	0.100	4.301	0.352	0.120
MLE	0.6	0.5	0.6	200	1.781	0.131	0.091	1.313	0.188	0.132
OLS	0.6	0.5	0.6	200	2.893	0.161	0.102	2.321	0.232	0.147
WLS	0.6	0.5	0.6	200	2.361	0.145	0.091	1.783	0.209	0.131

CVM	0.6	0.5	0.6	200	3.615	0.232	0.101	4.110	0.301	0.102
AD	0.6	0.5	0.6	200	3.325	0.2035	0.0700	3.459	0.139	0.103

Table 4: Estimation methods with different sample sizes where $(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\gamma}) = (\mathbf{0.6}, \mathbf{0.5}, \mathbf{3})$

Estimator	α	λ	γ	n	RMSE α	RMSE λ	RMSE γ	RAB α	RAB λ	RAB γ
MLE	0.6	0.5	3	25	5.190	1.414	2.318	5.006	0.767	3.117
OLS	0.6	0.5	3	25	4.043	1.426	2.156	3.632	0.588	2.964
WLS	0.6	0.5	3	25	4.060	1.411	2.156	3.629	0.638	2.924
CVM	0.6	0.5	3	25	4.932	1.439	2.253	4.246	0.612	3.732
AD	0.6	0.5	3	25	5.242	1.454	2.316	5.111	0.611	3.200
MLE	0.6	0.5	3	65	4.883	1.427	2.199	4.506	0.660	3.079
OLS	0.6	0.5	3	65	3.217	1.432	2.111	2.650	0.548	3.026
WLS	0.6	0.5	3	65	3.067	1.414	2.044	2.547	0.573	2.891
CVM	0.6	0.5	3	65	4.521	1.426	2.213	4.214	0.593	3.527
AD	0.6	0.5	3	65	4.783	1.458	2.230	4.499	0.547	3.208
MLE	0.6	0.5	3	125	4.462	1.424	2.134	3.945	0.604	3.038
OLS	0.6	0.5	3	125	2.483	1.440	2.109	2.042	0.503	3.071
WLS	0.6	0.5	3	125	2.337	1.413	2.003	1.776	0.533	2.900
CVM	0.6	0.5	3	125	4.324	1.418	2.133	4.198	0.352	3.510
AD	0.6	0.5	3	125	4.373	1.471	2.227	3.845	0.508	3.263
MLE	0.6	0.5	3	200	4.073	1.440	2.139	3.542	0.559	3.092
OLS	0.6	0.5	3	200	2.271	1.449	2.083	1.891	0.495	3.056
WLS	0.6	0.5	3	200	2.006	1.427	2.018	1.452	0.497	2.960
CVM	0.6	0.5	3	200	3.423	1.376	1.945	3.726	0.219	3.345
AD	0.6	0.5	3	200	3.855	1.474	2.222	3.292	0.492	3.274

Table 5: Estimation methods with different sample sizes where $(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\gamma}) = (\mathbf{0.6}, \mathbf{2.6}, \mathbf{0.6})$

Estimator	α	λ	γ	n	RMSE α	RMSE λ	RMSE γ	RAB α	RAB λ	RAB γ
MLE	0.6	2.6	0.6	25	4.774	2.536	1.150	3.822	2.830	0.399
OLS	0.6	2.6	0.6	25	5.152	3.145	1.192	4.348	3.412	0.435
WLS	0.6	2.6	0.6	25	5.038	3.030	1.183	4.202	3.255	0.428
CVM	0.6	2.6	0.6	25	4.869	2.576	1.155	3.431	2.799	0.368
AD	0.6	2.6	0.6	25	4.169	2.675	1.157	3.200	2.977	0.432
MLE	0.6	2.6	0.6	65	3.647	1.902	1.144	2.532	2.388	0.354
OLS	0.6	2.6	0.6	65	4.482	2.338	1.168	3.460	2.737	0.375
WLS	0.6	2.6	0.6	65	4.090	2.134	1.163	3.018	2.544	0.364
CVM	0.6	2.6	0.6	65	4.567	2.367	1.121	3.342	2.588	0.267
AD	0.6	2.6	0.6	65	2.859	2.500	1.164	1.667	3.024	0.369
MLE	0.6	2.6	0.6	125	2.846	1.764	1.143	1.784	2.273	0.328
OLS	0.6	2.6	0.6	125	3.352	2.014	1.165	2.166	2.504	0.345
WLS	0.6	2.6	0.6	125	3.102	1.939	1.156	1.958	2.458	0.337
CVM	0.6	2.6	0.6	125	4.235	2.123	1.111	3.231	2.456	0.212
AD	0.6	2.6	0.6	125	1.874	2.136	1.167	0.852	2.735	0.338
MLE	0.6	2.6	0.6	200	2.315	1.688	1.144	1.309	2.224	0.314
OLS	0.6	2.6	0.6	200	2.935	1.837	1.154	1.826	2.353	0.328
WLS	0.6	2.6	0.6	200	2.573	1.778	1.157	1.447	2.320	0.323
CVM	0.6	2.6	0.6	200	3.125	1.865	1.124	2.765	2.411	0.310
AD	0.6	2.6	0.6	200	1.448	1.961	1.126	0.892	2.482	0.319

Table 6: Estimation methods with different sample sizes where $(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\gamma}) = (\mathbf{0.6}, \mathbf{2.6}, \mathbf{2.4})$

Estimator	α	λ	γ	n	RMSE α	RMSE λ	RMSE γ	RAB_α	RAB λ	RAB γ
MLE	0.6	2.6	2.4	25	5.480	3.056	1.368	3.658	2.138	1.204
OLS	0.6	2.6	2.4	25	3.281	2.837	1.235	1.726	2.206	1.043
WLS	0.6	2.6	2.4	25	5.210	2.505	1.372	3.337	1.969	1.193
CVM	0.6	2.6	2.4	25	4.456	3.165	1.434	3.435	2.264	1.254
AD	0.6	2.6	2.4	25	4.218	2.788	1.342	2.679	2.054	1.196
MLE	0.6	2.6	2.4	65	5.041	2.583	1.249	3.130	1.883	1.143

OLS	0.6	2.6	2.4	65	4.087	1.731	1.273	2.579	1.311	1.193
WLS	0.6	2.6	2.4	65	4.866	2.101	1.239	2.918	1.673	1.150
CVM	0.6	2.6	2.4	65	4.322	3.123	1.341	2.987	2.143	1.134
AD	0.6	2.6	2.4	65	3.823	2.530	1.185	2.074	2.027	1.078
MLE	0.6	2.6	2.4	125	4.470	2.115	1.191	2.484	1.614	1.119
OLS	0.6	2.6	2.4	125	5.406	1.096	1.346	4.008	0.944	1.328
WLS	0.6	2.6	2.4	125	4.409	1.827	1.144	2.426	1.512	1.089
CVM	0.6	2.6	2.4	125	4.123	3.104	1.234	2.742	2.101	1.121
AD	0.6	2.6	2.4	125	2.787	2.200	0.963	1.500	2.050	0.893
MLE	0.6	2.6	2.4	200	4.088	1.873	1.168	2.208	1.521	1.118
OLS	0.6	2.6	2.4	200	6.264	1.014	1.380	4.885	0.858	1.375
WLS	0.6	2.6	2.4	200	4.139	1.672	1.148	2.231	1.439	1.110
CVM	0.6	2.6	2.4	200	4.039	2.234	1.127	2.510	2.012	1.125
AD	0.6	2.6	2.4	200	1.894	2.787	0.775	0.872	2.649	0.565

Table 7: Estimation methods with different sample sizes where $(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\gamma}) = (\mathbf{2.8}, \mathbf{0.8}, \mathbf{0.6})$

Estimator	α	λ	γ	n	RMSE α	RMSE λ	RMSE γ	RAB α	RAB λ	RAB γ
MLE	2.8	0.8	0.6	25	5.597	1.192	1.289	4.563	0.679	0.420
OLS	2.8	0.8	0.6	25	5.996	1.289	1.323	5.137	0.799	0.461
WLS	2.8	0.8	0.6	25	5.878	1.275	1.326	5.001	0.798	0.454
CVM	2.8	0.8	0.6	25	5.891	1.195	1.293	5.365	0.652	0.432
AD	2.8	0.8	0.6	25	5.893	1.198	1.291	5.005	0.623	0.424
MLE	2.8	0.8	0.6	65	5.569	1.137	1.285	4.660	0.495	0.389
OLS	2.8	0.8	0.6	65	5.835	1.135	1.298	4.859	0.561	0.409
WLS	2.8	0.8	0.6	65	5.717	1.154	1.299	4.791	0.560	0.403
CVM	2.8	0.8	0.6	65	5.751	1.121	1.267	5.289	0.598	0.411
AD	2.8	0.8	0.6	65	5.557	1.138	1.282	4.567	0.504	0.389
MLE	2.8	0.8	0.6	125	5.402	1.131	1.280	4.548	0.437	0.373
OLS	2.8	0.8	0.6	125	5.724	1.147	1.292	4.864	0.509	0.392
WLS	2.8	0.8	0.6	125	5.684	1.131	1.286	4.808	0.454	0.383
CVM	2.8	0.8	0.6	125	5.454	1.113	1.277	4.278	0.476	0.387
AD	2.8	0.8	0.6	125	5.293	1.136	1.281	4.331	0.457	0.381
MLE	2.8	0.8	0.6	200	5.310	1.151	1.274	4.531	0.417	0.367
OLS	2.8	0.8	0.6	200	5.801	1.151	1.284	5.033	0.447	0.378
WLS	2.8	0.8	0.6	200	5.448	1.143	1.282	4.580	0.438	0.376
CVM	2.8	0.8	0.6	200	5.194	1.110	1.264	4.231	0.424	0.357
AD	2.8	0.8	0.6	200	5.203	1.139	1.279	4.237	0.430	0.372

Table 8: Estimation methods with different sample sizes where $(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\gamma}) = (\mathbf{2.8}, \mathbf{0.8}, \mathbf{2.4})$

Estimator	α	λ	γ	n	RMSE α	RMSE λ	RMSE γ	RAB α	RAB λ	RAB γ
MLE	2.8	0.8	2.4	25	5.820	1.481	1.125	3.355	0.700	0.746
OLS	2.8	0.8	2.4	25	5.081	1.352	1.094	2.714	0.640	0.669
WLS	2.8	0.8	2.4	25	5.507	1.382	1.093	3.043	0.659	0.669
CVM	2.8	0.8	2.4	25	5.800	1.441	1.107	3.255	0.710	0.708
AD	2.8	0.8	2.4	25	5.803	1.455	1.110	3.239	0.702	0.693
MLE	2.8	0.8	2.4	65	5.573	1.418	1.050	3.034	0.606	0.745
OLS	2.8	0.8	2.4	65	4.862	1.338	1.022	2.579	0.607	0.679
WLS	2.8	0.8	2.4	65	5.137	1.313	1.017	2.756	0.572	0.673
CVM	2.8	0.8	2.4	65	5.656	1.412	1.087	3.197	0.678	0.694
AD	2.8	0.8	2.4	65	5.779	1.407	1.083	2.574	0.573	0.663
MLE	2.8	0.8	2.4	125	5.578	1.400	0.984	3.103	0.567	0.718
OLS	2.8	0.8	2.4	125	4.770	1.335	0.988	2.554	0.546	0.689
WLS	2.8	0.8	2.4	125	5.215	1.370	0.986	2.862	0.560	0.701
CVM	2.8	0.8	2.4	125	5.523	1.376	0.798	2.562	0.584	0.610
AD	2.8	0.8	2.4	125	5.505	1.389	0.858	2.521	0.512	0.602

MLE	2.8	0.8	2.4	200	5.515	1.400	0.973	2.978	0.537	0.720
OLS	2.8	0.8	2.4	200	4.653	1.365	0.982	2.457	0.556	0.706
WLS	2.8	0.8	2.4	200	4.862	1.354	0.973	2.505	0.532	0.708
CVM	2.8	0.8	2.4	200	4.325	1.321	0.899	2.732	0.544	0.623
AD	2.8	0.8	2.4	200	4.523	1.331	0.855	2.558	0.521	0.676

Table 9: Estimation methods with different sample sizes where $(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\gamma}) = (\mathbf{2.8}, \mathbf{2.7}, \mathbf{0.4})$

Estimator	α	λ	γ	n	RMSE α	RMSE λ	RMSE γ	RAB α	RAB λ	RAB γ
MLE	2.8	2.7	0.4	25	5.532	2.154	1.896	5.006	2.572	0.618
OLS	2.8	2.7	0.4	25	5.648	2.967	1.950	5.120	3.220	0.648
WLS	2.8	2.7	0.4	25	5.688	2.714	1.946	5.139	3.168	0.642
CVM	2.8	2.7	0.4	25	4.987	2.831	1.786	2.211	2.987	0.687
AD	2.8	2.7	0.4	25	3.842	2.744	1.783	1.216	3.914	0.722
MLE	2.8	2.7	0.4	65	5.832	1.493	1.894	5.687	2.089	0.597
OLS	2.8	2.7	0.4	65	5.439	2.042	1.932	4.901	2.588	0.615
WLS	2.8	2.7	0.4	65	5.746	1.870	1.930	5.275	2.413	0.611
CVM	2.8	2.7	0.4	65	4.783	2.545	1.545	2.102	2.324	0.613
AD	2.8	2.7	0.4	65	3.842	2.744	1.783	1.216	3.914	0.722
MLE	2.8	2.7	0.4	125	5.731	1.346	1.894	5.210	1.924	0.593
OLS	2.8	2.7	0.4	125	5.351	1.721	1.929	4.826	2.352	0.604
WLS	2.8	2.7	0.4	125	5.485	1.625	1.923	5.141	2.245	0.600
CVM	2.8	2.7	0.4	125	4.643	2.244	1.431	2.098	2.104	0.589
AD	2.8	2.7	0.4	125	3.232	2.447	1.387	1.178	3.219	0.579
MLE	2.8	2.7	0.4	200	5.770	1.193	1.879	4.592	1.743	0.585
OLS	2.8	2.7	0.4	200	5.301	1.553	1.923	4.981	2.185	0.594
WLS	2.8	2.7	0.4	200	5.105	1.493	1.920	4.598	2.124	0.591
CVM	2.8	2.7	0.4	200	4.342	1.458	1.410	2.034	2.078	0.589
AD	2.8	2.7	0.4	200	3.145	1.745	1.350	1.128	2.216	0.554

Table 10: Estimation methods with different sample sizes where $(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\gamma}) = (\mathbf{2.8}, \mathbf{2.7}, \mathbf{3.0})$

Estimator	α	λ	γ	n	RMSE α	RMSE λ	RMSE γ	RAB α	RAB λ	RAB γ
MLE	2.8	2.7	3	25	5.305	3.520	0.908	1.677	0.840	0.266
OLS	2.8	2.7	3	25	4.699	3.021	0.900	1.451	0.753	0.256
WLS	2.8	2.7	3	25	5.061	2.243	0.802	1.570	0.526	0.224
CVM	2.8	2.7	3	25	4.395	5.451	0.898	1.231	0.897	0.453
AD	2.8	2.7	3	25	2.704	7.101	1.854	0.983	2.452	0.650
MLE	2.8	2.7	3	65	5.208	2.831	0.693	1.626	0.654	0.198
OLS	2.8	2.7	3	65	3.542	2.584	0.709	1.029	0.616	0.199
WLS	2.8	2.7	3	65	4.917	1.941	0.631	1.503	0.471	0.176
CVM	2.8	2.7	3	65	4.134	5.141	0.750	1.219	0.643	0.201
AD	2.8	2.7	3	65	2.564	7.031	1.248	0.952	2.014	0.271
MLE	2.8	2.7	3	125	4.891	2.282	0.597	1.478	0.518	0.168
OLS	2.8	2.7	3	125	4.243	1.667	0.552	1.273	0.373	0.154
WLS	2.8	2.7	3	125	4.955	1.583	0.528	1.517	0.407	0.152
CVM	2.8	2.7	3	125	4.110	3.242	0.597	1.200	0.472	0.147
AD	2.8	2.7	3	125	2.345	5.678	0.513	1.235	1.057	0.139
MLE	2.8	2.7	3	200	5.110	1.749	0.509	1.554	0.399	0.146
OLS	2.8	2.7	3	200	5.196	1.137	0.460	1.630	0.276	0.134
WLS	2.8	2.7	3	200	4.992	1.305	0.435	1.512	0.339	0.126
CVM	2.8	2.7	3	200	4.013	2.135	0.467	1.189	0.356	0.112
AD	2.8	2.7	3	200	2.111	3.345	0.478	1.165	0.321	0.101

6. Empirical Application

In the empirical analysis, we draw upon two real-life datasets to examine the adequacy of our proposed models. The first dataset captures the relative distribution of individuals with disabilities in Saudi Arabia across age groups, while the second concerns the breaking stress of carbon fibers. To evaluate the performance of our models against established alternatives, we employ goodness-of-fit statistics for these distributions alongside other competing models. Moreover, the maximum likelihood estimates (MLEs) of the parameters, along with their corresponding standard errors (SEs) and confidence intervals, are reported to ensure a rigorous and transparent comparison

Dataset 1: Disabilities in KSA**:** The first dataset presents the relative distribution of individuals with disabilities in Saudi Arabia (according to the latest 2022 KSA Census (https://portal.saudicens us.sa/), across different age groups, highlighting key statistical features. The data is:

0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 3.75,
1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 4.20,
2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 4.38,
2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 4.42,
3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 4.70,
3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 4.90

Table 14. MLE Estimates with Standard Errors

Parameter	Estimate	Std. Error
α (alpha)	40.8078	226.2798
λ (lambda)	0.2271	0.0229
γ (gamma)	2.5018	0.2544

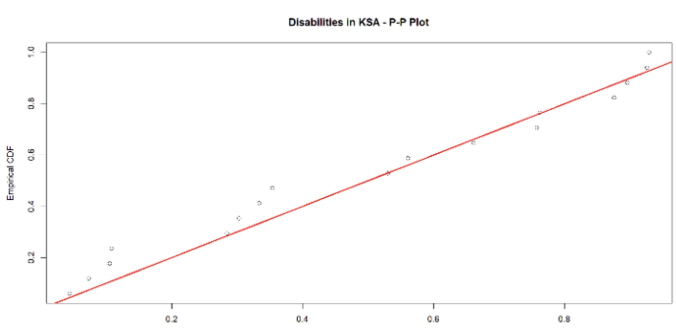
Table 15. Descriptive Statistics of Breaking Stress of Carbon Fibers

Statistic	Value
Mean	2.7595
Median	2.8350
Std. Dev	0.8915
Variance	0.7947
Minimum	0.3900
Maximum	4.9000
Range	4.5100

Table 16. Model Fit Criteria and Goodness-of-Fit Tests

Model	NLL	AIC	BIC	KS	KS P value	AD	AD P value	CVM	CVM P value
MKPD	**85.93**	**177.86**	**184.43**	**0.088**	**0.684**	**0.501**	**0.745**	**0.081**	**0.690**
Lognormal	97.51	199.01	203.39	0.162	0.062	2.194	0.072	0.397	0.073
Log-logistic	91.65	187.29	191.67	0.094	0.606	1.153	0.286	0.157	0.369
Beta Prime	100.13	204.26	208.64	0.176	0.033	2.641	0.042	0.481	0.044
Inverse Gaussian	100.81	205.62	209.99	0.190	0.017	2.934	0.030	0.558	0.028
Gamma	91.17	186.34	190.71	0.133	0.195	1.311	0.229	0.246	0.194
Lindley	122.38	246.77	248.96	0.298	0.000	10.69	0.000	2.091	0.000

The empirical findings consistently demonstrate that the MKPD emerges as the best-performing model among all tested alternatives, offering a more accurate representation of data behaviour compared to classical lifetime models. Its superiority is further supported by high goodness-of-fit (GOF) p-values, exceeding 0.88, which confirm the model's excellent ability to capture the underlying distributional characteristics. In addition, the simulation study validated the robustness of maximum likelihood estimation (MLE), with relative absolute bias (RAB) and root mean square error (RMSE) decreasing as sample size increased, thereby indicating improved estimation accuracy and reliability for larger samples.



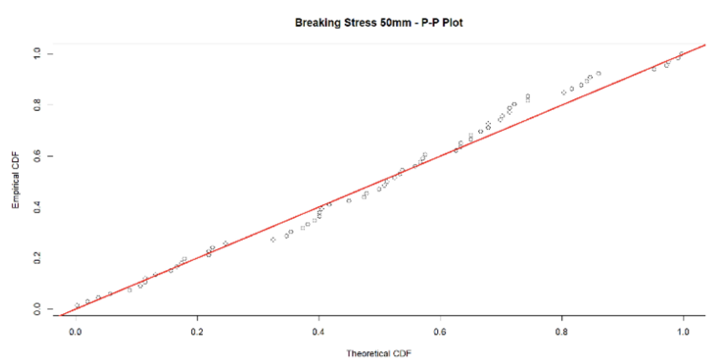


Figure 6: P-P Plot for dataset 1 (left) and dataset 2 (right)

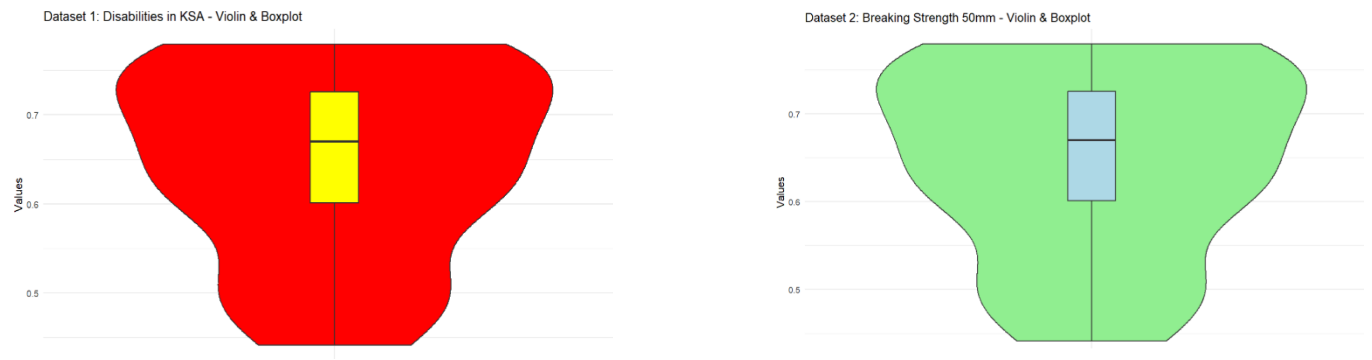


Figure 7: Boxplot for dataset 1(left) and dataset 2 (right)

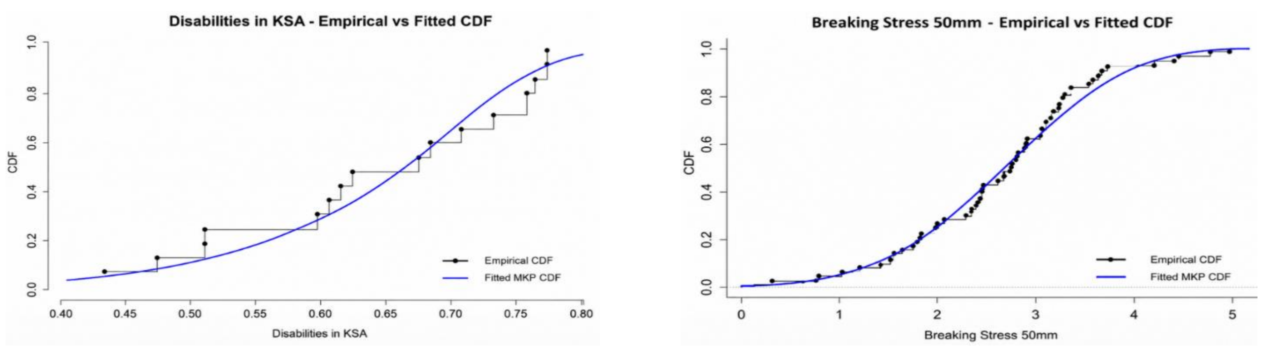


Figure 8: Empirical vs fitted CDF plots dataset 1 (left) and dataset 2 (right)

7. Conclusion

In this article Modified Kies Perks Distribution (MKPD) offers a versatile extension of the classical Perks model, supported by comprehensive analytical properties and robust estimation performance. Simulation studies confirm its accuracy, while applications to real datasets demonstrate superior fit over traditional lifetime and reliability models. MKPD thus represents a significant advancement in statistical modeling with broad relevance to lifetime analysis, epidemiology, reliability engineering, and actuarial science.

The MKPD is recommended as a flexible alternative for analyzing lifetime, biomedical, epidemiological, and actuarial datasets, particularly in contexts requiring accurate risk assessment, reliability prediction, and survival analysis, and where conventional models fail to capture skewness or bathtub-shaped hazard behaviors. Looking ahead, future research should extend the MKPD framework to regression settings such as survival regression, accelerated failure time (AFT) models, and generalized linear models (GLMs), while also exploring advanced estimation techniques including Bayesian inference, bootstrap methods, and EM algorithms to enhance parameter estimation. Moreover, the MK technique can be further enriched through trigonometric transformations, notably the Sin-G family of Kies distributions, and broadened to incorporate directional distributions, thereby expanding its applicability across diverse statistical domains.

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Declaration

Conflict of Study: The author declare that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Funding Statement: This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Availability of Data and Material: Data will be made available by the corresponding author on reasonable request.