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A New Reliability Model Based on Weibull Distribution

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ABSTRACT

We examine the mean remaining strength of a parallel system under the stress-strength framework, utilizing the Weibull distribution. This distribution is selected due to its modeling flexibility, ability to represent symmetric data, and widespread application in reliability analysis, engineering, and life testing. The likelihood ratio ordering of the system's remaining strength is analyzed for a two-component case. Both the maximum likelihood estimator and Bayesian estimator for the mean remaining strength are derived. Additionally, we develop both asymptotic and bootstrap confidence intervals for the mean remaining strength. A simulation study is conducted to evaluate and compare the performance of these estimators. The methodology is further illustrated using a real dataset related to breast cancer.

Keywords*:* Mean remaining strength (MRS); Weibull distribution; Asymptotic Confidence Interval (A.C.I.); Bootstrap interval; Bayes estimator; Monte-Carlo simulation

1. Introduction

The The stress-strength model characterizes system reliability by considering both stress and strength as random variables. Specifically, reliability is expressed as $R = P[Y < X]$ where Y represents stress and X a denotes strength. A system failure occurs when the applied stress surpasses the component's strength. Numerous researchers have explored this model using various probability distributions, beginning with early foundational studies (1) and for more past references see (2). Recently investigated by (3), (4), (5), (6) and (7). The Weibull distribution is among the most applied models in reliability analysis. Due to its versatility, it has attracted considerable attention from many researchers, including authors such as (8) when investigated stress-strength model of Weibull distribution based on progressively censored samples. (9) investigated different estimation methods of stress-strength model for Weibull distribution. Also, (10) investigated the stress-strength model for Weibull distribution with different shape parameters. For more details of Weibull distribution see (11), (12). (13)defined the MRS of a component is described as the expected residual strength given that the component has not failed under stress, expressed as (i.e., $\psi = E[X - Y|Y < X]$). This measure helps estimate how much longer, on average, a component can continue to function safely when exposed to stress. (13), the mean remaining strength (MRS) within basic stress-strength frameworks, as well as in K-out-of-n:F parallel and series systems, assuming that stress and strength are independent random variables. (14) studied MRS for exchangeable strength components under the common stress. (15) investigated dependent strength components which are subject to a common stress. (16) investigated the MRS of parallel system in strength model baes on exponential distribution. (17) investigated MRS estimation of multi-state system based on nonparametric Bayesian method. In this paper, we investigate the MRS based on Weibull distribution because it is an important distribution in reliability applications.

The main contribution of this study is the investigation of the mean remaining strength of a parallel system within a stress–strength framework when the component strengths and stresses follow Weibull distributions. Although stress–strength reliability has been extensively studied for individual components and several system configurations, the mean remaining strength of a parallel system introduces an additional level of difficulty because the system strength is determined by the maximum of the component strengths. Consequently, its distribution is not obtained by directly applying the corresponding result for a single component. In addition, the remaining-strength quantity involves conditional expectations and therefore requires the joint treatment of the system-strength distribution and the stress distribution.

The Weibull model is particularly relevant because its shape and scale parameters provide considerable flexibility in representing increasing, decreasing, and constant failure-rate behavior. Thus, results obtained for exponential or other special distributions cannot, in general, be directly transferred to the Weibull setting. The proposed analysis derives the corresponding reliability and mean-remaining-strength measures specifically for the Weibull parallel system and develops classical and Bayesian inferential procedures for the associated parameters.

The novelty of the present work therefore lies not merely in adopting the Weibull distribution, but in the combined treatment of a parallel-system stress–strength structure, mean remaining strength, stochastic ordering, and classical and Bayesian inference within a unified Weibull framework. This provides a broader assessment of system performance than conventional stress–strength reliability alone and extends existing methodology to a practically important class of reliability models.

A Weibull distribution with shape parameter α and scale parameter β will be denoted by $WE(\alpha, \beta)$ are given by as

$$f(x) = \alpha\, \beta\, x^{\alpha-1} \exp[-\beta\, x^{\alpha}] \qquad x > 0, \alpha, \beta > 0, \tag{1}$$

and

$$F(x) = 1 - exp[-\beta\, x^{\alpha}] \qquad x > 0, \alpha, \beta > 0. \tag{2}$$

Section 2 focuses on deriving the mean remaining strength (MRS) for a parallel system within the stress-strength framework, if $X_1 \dots \dots X_{n_1}$ and $Y_1 \dots \dots Y_{n_2}$ are independent and identically distributed Weibull random variables representing strength and stress, respectively. Both sets share a common shape parameter α but differ in scale parameters $\beta_1,\ \beta_2$ for strength and stress respectively. (i. e. $X \sim WE(\alpha, \beta_1)$ and $Y \sim WE(\alpha, \beta_2)$). In the present study, the populations of strength and stress are assumed to follow Weibull distributions with a common shape parameter α, while their scale parameters are allowed to differ. This assumption has both practical and theoretical motivations. Practically, the Weibull shape parameter determines the general form of failure-rate behavior and may reasonably be assumed to be common when the stress and strength processes are governed by similar physical, environmental, or degradation mechanisms. The different scale parameters then allow the two populations to operate at different characteristic levels, reflecting differences in the magnitude of the applied stress and the resistance or capacity of the components. This section also includes a stochastic comparison of the MRS for two-component parallel systems. Section 3 provides the estimation methods for MRS, along with the development of both asymptotic and bootstrap confidence intervals. In Section 4, a simulation study is carried out to assess and compare the performance of the proposed estimators. Section 5 presents a real-world case study to demonstrate the practical application of the model. The conclusions are summarized in Section 6.

2. Model Description

In this section, first, derive the MRS of parallel system based on the Weibull distribution. Second, introduce the stochastic ordering result of our system.

2.1 Derive MRS

Let $X_1 \dots \dots X_{n_1}$ be independent strength variables, and $Y_1 \dots \dots Y_{n_2}$ be independent stress variables, follow the Weibull distribution with common shape parameter α and different scale parameters β_1, β_2 respectively. (i. e. $X \sim WE(\alpha, \beta_1)$ and $Y \sim WE(\alpha, \beta_2)$). To get the stress- strength model of parallel system is given by,

$$R_{n_1,n_2} = P\left[Y_{n_2:n_2} < X_{n_1:n_1}\right].$$

Let $X_{1:n_1} \le \dots \le X_{n_1:n_1}$ be ordered strength of components and $Y_{1:n_2} \le \dots \le Y_{n_2:n_2}$ be ordered stress of the components. Then

$$R_{n_1,n_2} = \int_0^{\infty} F_{Y_{n_2:n_2}}(x)\, f_{X_{n_1:n_1}}(x)\ dx,$$

where, $$f_{X_{n_1:n_1}}(x) = n_1[\alpha\, \beta_1\, x^{\alpha-1} \exp[-\beta_1\, x^{\alpha}]\ ][1 - \exp[-\beta_1\, x^{\alpha}]\qquad]^{n_1-1},$$

and $$F_{Y_{n_2:n_2}}(x) = [1 - exp[-\beta_2\, x^{\alpha}]]^{n_2},$$

$$R_{n_1,n_2} = \int_0^{\infty} [1 - exp[-\beta_2\, x^{\alpha}]]^{n_2}\, n_1[\alpha\, \beta_1\, x^{\alpha-1} exp[-\ \beta_1\, x^{\alpha}]][1 - exp[-\beta_1\, x^{\alpha}]]^{n_1-1}\, dx,$$

Using binomial theory $(1-z)^n = \sum_{j=0}^{n} \binom{n}{j} (-1)^j z^j$. Setting $z = \exp[-\beta\, x^{\alpha}]$

and integration, we get

Then, $$R_{n_1,n_2} = n_1\, \beta_1\, \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2} \frac{(-1)^{i+j} \binom{n_1-1}{i} \binom{n_2}{j}}{(i+1)\, \beta_1 + j\, \beta_2},$$

Hence, the MRS of the parallel system when the stress $Y_{n_2:n_2}$ is applied is $\psi_{n_1:n_2} = E[X_{n_1:n_1} - Y_{n_2:n_2} | Y_{n_2:n_2} < X_{n_1:n_1}]$ and the cumulative of the conditional random variable $Z \cong X_{n_1:n_1} - Y_{n_2:n_2} | Y_{n_2:n_2} < X_{n_1:n_1}$ is defined as,

$$F_Z(x) = P[X_{n_1:n_1} - Y_{n_2:n_2} \le x | Y_{n_2:n_2} < X_{n_1:n_1}]$$
$$= \frac{P[X_{n_1:n_1} \le Y_{n_2:n_2} + x | Y_{n_2:n_2} < X_{n_1:n_1}\ ]}{P[|Y_{n_2:n_2} < X_{n_1:n_1}]}$$

$$= \frac{1}{R_{n_1,n_2}} \; P[X_{n_1:n_1} \leq Y_{n_2:n_2} + x \big| Y_{n_2:n_2} < X_{n_1:n_1}].$$

When $Y_{n_2:n_2} = y$,

$$P\big[X_{n_1:n_1} \leq Y_{n_2:n_2} + x \big| Y_{n_2:n_2} < X_{n_1:n_1}\big] = \int_0^\infty P[y < X_{n_1:n_1} < y + x] \; d\, F_Y(y)$$

$$= \int_0^\infty \Big[F_{X_{n_1:n_1}}(y+x) - F_{X_{n_1:n_1}}(y)\Big] \; d\, F_Y(y).$$

Using binomial theory $(1-z)^n = \sum_{j=0}^{n} \binom{n}{j} (-1)^j z^j$. Setting $z = \exp[-\beta\, x^\alpha]$ and integration, we get

$$P\big[X_{n_1:n_1} \leq Y_{n_2:n_2} + x \big| Y_{n_2:n_2} < X_{n_1:n_1}\big] = n_2\, \beta_2 \sum_{i=0}^{n_1} \sum_{j=0}^{n_2-1} \frac{(-1)^{i+j} \binom{n_1}{i} \binom{n_2-1}{j} (e^{-i\,\beta_1\, x^\alpha} - 1)}{i\,\beta_1 + (j+1)\beta_2}.$$

Hence,
$$F_Z(x) = \frac{n_2\, \beta_2}{R_{n_1,n_2}} \sum_{i=0}^{n_1} \sum_{j=0}^{n_2-1} \frac{(-1)^{i+j} \binom{n_1}{i} \binom{n_2-1}{j} (e^{-i\,\beta_1\, x^\alpha} - 1)}{i\,\beta_1 + (j+1)\beta_2}.$$

And the pdf of the random Z is,

$$f_Z(x) = \frac{d\, F_Z(x)}{dx} = \frac{n_2\, \beta_2\, \alpha\, \beta_1}{R_{n_1,n_2}} \sum_{i=1}^{n_1} \sum_{j=0}^{n_2-1} \frac{(-1)^{i+j+1} \binom{n_1}{i} \binom{n_2-1}{j} \; i\, x^{\alpha-1} \cdot \exp[-i\, \beta_1\, x^\alpha]}{i\,\beta_1 + (j+1)\beta_2}.$$

Then, $\psi_{n_1,n_2} = \int_0^\infty x\, f_Z(x)\, dx$

$$= \frac{n_2\, \beta_2\, \alpha\, \beta_1}{R_{n_1,n_2}} \sum_{i=1}^{n_1} \sum_{j=0}^{n_2-1} \int_0^\infty \frac{(-1)^{i+j+1} \binom{n_1}{i} \binom{n_2-1}{j} \; i\, x^{\alpha} \cdot \exp[-i\, \beta_1\, x^\alpha]}{i\,\beta_1 + (j+1)\beta_2} dx,$$

Using binomial theory $(1-z)^n = \sum_{j=0}^{n} \binom{n}{j} (-1)^j z^j$. Setting $z = \exp[-\beta\, x^\alpha]$ and integration, we get

$$\psi_{n_1,n_2} = \frac{n_2\, \beta_2\, \Gamma(1/\alpha)}{(\beta_1)^{\frac{1}{\alpha}}\, \alpha\, R_{n_1,n_2}} \sum_{i=1}^{n_1} \sum_{j=0}^{n_2-1} \frac{(-1)^{i+j+1} \binom{n_1}{i} \binom{n_2-1}{j}}{[i\,\beta_1 + (j+1)\,\beta_2]^{1/i\,\alpha}}. \tag{3}$$

Note that if $\alpha = 1$, we get the results of exponential distribution which derived by (16).

2.2 Stochastic Ordering Results

(18), (19) and (20) are investigated the likelihood ratio ordering results with the residual life of a random variable X at time θ which is defined as $X_\theta = [X - \theta | X > \theta]$. Also, for stochastic order relations among parallel systems from Weibull distribution see (21).

Definition:

Let X and Y be two lifetime random variables with probability density function, $f(x)$ and $g(y)$, respectively. X is said to be, smaller than Y in the likelihood ratio order denoted by $X \leq_{lr} Y$, if $\frac{g(x)}{f(x)}$ is increasing in x, for all x.

Since the random variable $Z \cong X_{n_1:n_1} - Y_{n_2:n_2} | Y_{n_2:n_2} < X_{n_1:n_1}$ is a special case of the residual life of a random variable X, then we will study the likelihood ratio ordering results with the MRS as follows: Let we consider two component parallel system ($n_1 = n_2 = 2$) then ,

Where,
$$f_Z(x) = \frac{2\,\beta_1\,\beta_2\,\alpha\, x^{\alpha-1}}{R_{2,2}} \left(\frac{2\exp[-\beta_1\, x^\alpha] + \exp[-2\,\beta_1\, x^\alpha]}{\beta_1 + \beta_2} - \frac{2\exp[-2\,\beta_1\, x^\alpha]}{2\,\beta_1 + \beta_2} - \frac{2\exp[-\beta_1\, x^\alpha]}{\beta_1 + 2\,\beta_2}\right),$$

where,
$$R_{2,2} = 1 - \frac{5\,\beta_1}{\beta_1 + \beta_2} + \frac{4\beta_1}{2\,\beta_1 + \beta_2} + \frac{2\,\beta_1}{\beta_1 + 2\,\beta_2}.$$

Theorem:

Suppose variables X_i , X_i^* are the strength and Y_i , Y_i^* are the stress variables with $X_i \cong WE(\alpha, \beta_1)$, $X_i^* \sim WE(\alpha, \beta_1^*)$, $Y_i \sim WE(\alpha, \beta_2)$, and $Y_i^* \sim WE(\alpha, \beta_2^*)$, $i = 1,2$. If $\beta_1^* < \beta_1$ and $\beta_2 < \beta_2^*$ then we have $Z \leq_{lr} Z^*$ where $Z \sim [X_{2:2} - Y_{2:2} | Y_{2:2} < X_{2:2}]$ and $Z^* \sim [X^*{}_{2:2} - Y^*{}_{2:2} | Y^*{}_{2:2} < X^*{}_{2:2}]$.

Proof: If we show $\frac{f_{Z^*}(x)}{f_Z(x)}$ is increasing in x the proof is complete see (21).

$$L(x) = \frac{f_{Z^*}(x)}{f_Z(x)} = \frac{\beta_1^*(7\beta_1 + 2\beta_2) e^{2(\beta_1 - \beta_1^*)x^\alpha} (\beta_1^* + 2\beta_2^* - 2(2\beta_1^* + \beta_2^*) e^{\beta_1^* x^\alpha})}{\beta_1 (7\beta_1^* + 2\beta_2^*)(\beta_1 + 2\beta_2 - 2(2\beta_1 + \beta_2) e^{\beta_1 x^\alpha})},$$

We have to show $\frac{dL(x)}{dx} \geq 0$, $\forall\, x \geq 0$. The derivative of $L(x)$ is.

$$\frac{dL(x)}{dx} =^{sign} \frac{2\alpha\beta_1^*(7\beta_1 + 2\beta_2) e^{2(\beta_1 - \beta_1^*)x^\alpha}}{\beta_1(7\beta_1^* + 2\beta_2^*)(\beta_1 + 2\beta_2 - 2(2\beta_1 + \beta_2)e^{\beta_1 x^\alpha})^2} *$$
$$\Big(-\beta_1^*(2\beta_1^* + \beta_2^*) e^{\beta_1^* x^\alpha} \big(\beta_1 + 2\beta_2 - 2(2\beta_1 + \beta_2) e^{\beta_1 x^\alpha}\big) + \beta_1 (2\beta_1 + \beta_2) e^{\beta_1 x^\alpha} \big(\beta_1^* + 2\beta_2^* - 2(2\beta_1^* + \beta_2^*) e^{\beta_1^* x^\alpha}\big)$$
$$+ (\beta_1 - \beta_2^*)\big(\beta_1 + 2\beta_2 - 2(2\beta_1 + \beta_2)e^{\beta_1 x^\alpha}\big)\big(\beta_1^* + 2\beta_2^* - 2(2\beta_1^* + \beta_2^*)e^{\beta_1^* x^\alpha}\big)\Big) x^{-1+\alpha}.$$

To $\frac{dL(x)}{dx} \geq 0$, then $e^{2(\beta_1-\beta_1^*)x^\alpha} \leftrightarrow \beta_1 > \beta_1^*$, and after some calculations, we get,

$$\frac{dL(x)}{dx} \geq \beta_1^*(2\beta_1^* + \beta_2^*) - \beta_1(2\beta_1 + \beta_2),$$

But $\beta_1 > \beta_1^*$, then $\frac{dL(x)}{dx} \geq (\beta_2^* - \beta_2)$. Hence $\frac{dL(x)}{dx} \geq 0 \leftrightarrow \beta_2^* > \beta_2$ the proof if complete.

Also, graphical approach to prove this theorem, (See, Figure (1) and Figure (2), the proof of the Theorem is completed).

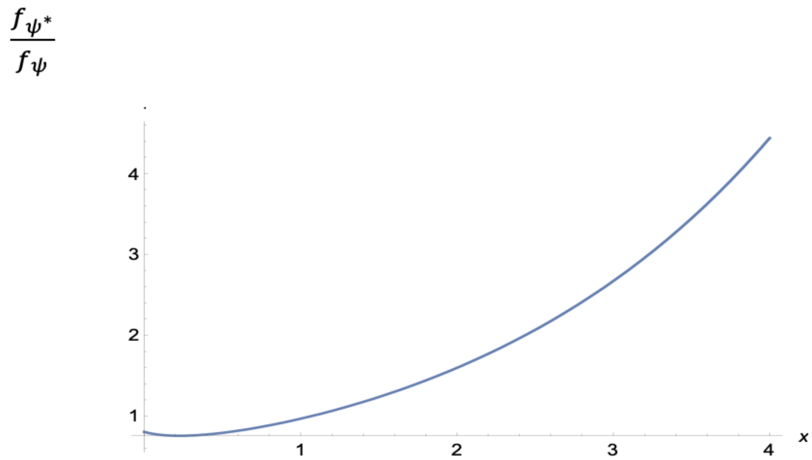

Figure 1: $n_1 = n_2 = 2$; $(\alpha, \beta_1) = (1,1.5)$, $(\alpha, \beta_1^*) = (1,1)$, $(\alpha, \beta_2) = (1,2)$, and $(\alpha, \beta_2^*) = (1,2.5)$.

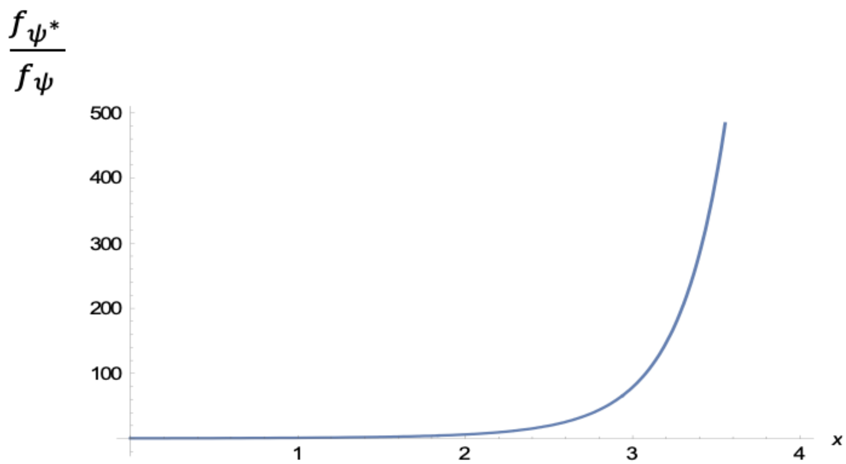

Figure 2: $n_1 = n_2 = 2$; $(\alpha, \beta_1) = (2,2.5)$, $(\alpha, \beta_1^*) = (2,2)$, $(\alpha, \beta_2) = (2,2)$, and $(\alpha, \beta_2^*) = (2,3.5)$.

3. Estimation of MRS of Parallel System ψ_{n_1,n_2}

The point and interval estimation of ψ_{n_1,n_2} are present. The maximum likelihood and Bayes estimators are presented. And asymptotic confidence interval and bootstrap interval are constructed.

3.1 Maximum Likelihood Estimator for ψ_{n_1,n_2}

Let the random variable of strength and stress of the parallel system be $V = X_{(n_1)}$, and $W = Y_{(n_2)}$,respectively. Let $V_1, \dots\dots, V_n$ and $W_1, \dots\dots, W_m$ be two random samples of size n and m respectively with the pdfs are defined as,

$$f(v) = n_1\, \alpha\, \beta_1\, v^{\alpha-1}\, e^{-\beta_1 v^\alpha} [1 - e^{-\beta_1 v^\alpha}]^{n_1-1} \quad \alpha\,,\ \beta_1 > 0\,, \quad v > 0.$$

And,

$$f(w) = n_2\, \alpha\, \beta_2\, w^{\alpha-1}\, e^{-\beta_2 w^\alpha} [1 - e^{-\beta_2 w^\alpha}]^{n_2-1} \quad \alpha\,,\ \beta_2 > 0\,, \quad w > 0.$$

Then, the likelihood function is.

$$L(\alpha\,, \beta_1, \beta_2) = \prod_{i=1}^{n} f(v_i) \prod_{j=1}^{m} f(w_j).$$

Now, the log-likelihood function is.

$$Ln\, L(\alpha\,, \beta_1, \beta_2) = c + (n+m)Ln[\alpha] + n\, Ln[\beta_1] + m\, Ln[\beta_2] - \beta_1 \sum_{i=1}^{n} v_i^\alpha$$
$$-\beta_2 \sum_{j=1}^{m} w_j^\alpha + (\alpha - 1)\sum_{i=1}^{n} Ln\,[v_i] + (\alpha - 1)\sum_{j=1}^{m} Ln\,[w_i]$$
$$+(n_1 - 1)\sum_{i=1}^{n} Ln[1 - e^{-\beta_1 v_i^\alpha}] + (n_2 - 1)\sum_{j=1}^{m} Ln[1 - e^{-\beta_2 w_j^\alpha}].$$

The maximum likelihood estimators $\alpha, \beta_1, and\ \beta_2$ denoted by $\hat{\alpha}$, $\hat{\beta}_1$ and $\hat{\beta}_2$ are the solution of the following nonlinear equations.

$$\frac{\partial Ln\, L(\alpha,\beta_1,\beta_2)}{\partial \alpha} = 0 = \frac{n+m}{\alpha} - \beta_1 \sum_{i=1}^{n} v_i^{\alpha}\ Ln[v_i] - \beta_2 \sum_{j=1}^{m} w_j^{\alpha}\ Ln[w_j] + \sum_{i=1}^{n} Ln\,[v_i] + \sum_{j=1}^{m} Ln\,[w_i] + (n_1 - 1) \sum_{i=1}^{n} \frac{\beta_1 v_i^{\alpha}\ Ln[v_i]\, e^{-\beta_1 v_i^{\alpha}}}{[1-e^{-\beta_1 v_i^{\alpha}}]} + +(n_2-1) \sum_{j=1}^{m} \frac{\beta_2 w_j^{\alpha}\ Ln[w_j]\, e^{-\beta_2 w_j^{\alpha}}}{[1-e^{-\beta_2 w_j^{\alpha}}]}.$$

$$\frac{\partial Ln\, L(\alpha,\beta_1,\beta_2)}{\partial \beta_1} = 0 = \frac{n}{\beta_1} - \sum_{i=1}^{n} v_i^{\alpha} + (n_1-1) \sum_{i=1}^{n} \frac{v_i^{\alpha}\ e^{-\beta_1 v_i^{\alpha}}}{[1-e^{-\beta_1 v_i^{\alpha}}]}.$$

$$\frac{\partial Ln\, L(\alpha,\beta_1,\beta_2)}{\partial \beta_2} = 0 = \frac{m}{\beta_2} - \sum_{j=1}^{m} w_j^{\alpha} + (n_2-1) \sum_{j=1}^{m} \frac{w_j^{\alpha}\ e^{-\beta_2 w_j^{\alpha}}}{[1-e^{-\beta_2 w_j^{\alpha}}]}.$$

Since we cannot obtain the closed form $\hat{\alpha}$, $\hat{\beta}_1$ and $\hat{\beta}_2$, then we used the numerical method to get them and using the invariance property of maximum likelihood estimators, the maximum likelihood estimator of $\psi_{\boldsymbol{n_1,n_2}}$ which denoted by $\hat{\psi}_{n_1,n_2}^{MLE}$,

$$\hat{\psi}_{n_1,n_2}^{MLE} = \frac{n_2\, \hat{\beta}_2\, \text{Gamma}(1/\hat{\alpha})}{(\hat{\beta}_1)^{1/\hat{\alpha}}\, \hat{\alpha}\, \hat{R}_{n_1,n_2}} \sum_{i=1}^{n_1} \sum_{j=0}^{n_2-1} \frac{(-1)^{i+j+1} \binom{n_1}{i} \binom{n_2-1}{j}}{[i\,\hat{\beta}_1 + (j+1)\hat{\beta}_2]^{1/i\hat{\alpha}}}. \tag{4}$$

and
$$\hat{R}_{n_1,n_2} = n_1\, \hat{\beta}_1 \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2} \frac{(-1)^{i+j} \binom{n_1-1}{i} \binom{n_2}{j}}{(i+1)\hat{\beta}_1 + j\,\hat{\beta}_2}.$$

3.2 Asymptotic Confidence Interval (AIC) for $\boldsymbol{\psi_{n_1,n_2}}$

To compute the (A.C.I), find the Fisher information matrix as

$$I(\Theta) = -\begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}.$$

Where, $\Theta = (\alpha, \beta_1, \beta_2)$. The derivatives used under information matrix are shown in as follows:

$$I_{11} = E\left[\frac{\partial^2 Ln\, L(\Theta)}{\partial \alpha^2}\right] = -\frac{n+m}{\alpha^2} + k_1 + k_2 + k_3 + k_4.$$

$$I_{22} = E\left[\frac{\partial^2 Ln\, L(\Theta)}{\partial \beta_1^2}\right] = -\frac{n}{\beta_1^2} - n_1\,(n_1-1) \sum_{i=1}^{n} \sum_{s_5=0}^{n_1-3} (-1)^{s_5} \binom{n_1-3}{s_5} \frac{2}{s_5^3\, \beta_1^2}.$$

$$I_{33} = E\left[\frac{\partial^2 Ln\, L(\Theta)}{\partial \beta_2^2}\right] = -\frac{m}{\beta_2^2} - n_2\,(n_2-1) \sum_{j=1}^{m} \sum_{s_6=0}^{n_2-3} (-1)^{s_6} \binom{n_2-3}{s_6} \frac{2}{s_6^3\, \beta_2^2}.$$

$$I_{12} = I_{21} = E\left[\frac{\partial^2 Ln\, L(\Theta)}{\partial \alpha\, \partial \beta_1}\right] = k_5 + k_6.$$

$$I_{13} = I_{31} = E\left[\frac{\partial^2 Ln\, L(\Theta)}{\partial \alpha\, \partial \beta_2}\right] = k_7 + k_8, \text{ and } I_{23} = I_{32} = 0.$$

where,

$$k_1 = -n_1 \sum_{i=1}^{n} \sum_{s_1=0}^{n_1-1} (-1)^{s_1} \binom{n_1-1}{s_1} \frac{6(-2+\gamma)\,\gamma + \pi^2 + 6\,\text{Log}[(1+s_1)\,\beta_1](-2+2\,\gamma + \text{Log}[(1+s_1)\beta_1])}{6\,\alpha^2\,(1+s_1)^2}.$$

$$k_2 = -n_2 \sum_{j=1}^{m} \sum_{s_2=0}^{n_2-1} (-1)^{s_2} \binom{n_2-1}{s_2} * \frac{6(-2+\gamma)\,\gamma + \pi^2 + 6\,\text{Log}[(1+s_2)\,\beta_2](-2+2\,\gamma + \text{Log}[(1+s_2)\beta_2])}{6\,\alpha^2\,(1+s_2)^2}.$$

$$k_3 = n_1(n_1-1) \sum_{i=1}^{n} \sum_{s_3=0}^{n_1-3} (-1)^{s_3} \binom{n_1-3}{s_3} - \frac{1}{6\,\alpha^2} \Big(\frac{2\big(6 + 6(-3+\gamma)\,\gamma + \pi^2 + 6\,\text{Log}[s_3\,\beta_1](-3 + 2\,\gamma + \text{Log}\,[s_3\,\beta_1])\big)}{s_3^3} - \frac{6(-2+\gamma)\gamma + \pi^2 + 6\text{Log}\,[s_3\,\beta_1]\,(-2 + 2\,\gamma + \text{Log}[s_3\,\beta_1])}{s_3^2} + \frac{6(-2+\gamma)\,\gamma + \pi^2 + 6\text{Log}\,[(1+s_3)\,\beta_1](-2+2\gamma + \text{Log}[(1+s_3)\,\beta_1])}{(1+s_3)^2}\Big).$$

$$k_4 = n_2(n_2-1) \sum_{j=1}^{m} \sum_{s_4=0}^{n_2-3} (-1)^{s_4} \binom{n_2-3}{s_4} \frac{1}{6\,\alpha^2}$$

$$\Big(\frac{2\big(6 + 6(-3+\gamma)\,\gamma + \pi^2 + 6\,\text{Log}[s_4\beta_2](-3 + 2\,\gamma + \text{Log}\,[s_4\,\beta_2])\big)}{s_4^3} - \frac{6(-2+\gamma)\gamma + \pi^2 + 6\text{Log}\,[s_4\,\beta_2]\,(-2 + 2\,\gamma + \text{Log}[s_4\,\beta_2])}{s_4^2} +$$

$$\frac{6(-2+\gamma)\,\gamma+\pi^2+6\text{Log}\,[(1+s_4)\,\beta_2](-2+2\gamma+\text{Log}[(1+s_4)\,\beta_2])}{(1+s_4)^2}).$$

$$k_5=\sum_{i=1}^{n}\sum_{s_7=0}^{n_1-1}(-1)^{s_7}\binom{n_1-1}{s_7}\left(\frac{-1+\gamma+\text{Log}[(1+i)\,\beta_1]}{(1+i)^2\alpha\,\beta_1}\right).$$

$$k_6=\sum_{i=1}^{n}\sum_{s_8=0}^{n_1-3}(-1)^{s_8}\binom{n_1-3}{s_8}\left(\frac{3+i(5+i)-\gamma(2+3i)+(-2+i)(1+i)^2\text{Log}[i\beta_1]-i^3\text{Log}[(1+i)\beta_1]}{i^3(1+i)^2\alpha\beta_1}\right).$$

$$k_7=\sum_{j=1}^{m}\sum_{s_9=0}^{n_2-1}(-1)^{s_9}\binom{n_2-1}{s_9}\left(\frac{-1+\gamma+\text{Log}[(1+j)\,\beta_2]}{(1+j)^2\alpha\,\beta_2}\right).$$

$$k_8=\sum_{j=1}^{m}\sum_{s_{10}=0}^{n_2-3}(-1)^{s_{10}}\binom{n_2-3}{s_{10}}\left(\frac{3+j(5+j)-\gamma(2+3j)+(-2+j)(1+j)^2\text{Log}[j\,\beta_3]-j^3\text{Log}\,[(1+j)\beta_2]}{j^3(1+j)^2\alpha\beta_2}\right).$$

Now, since $\hat{\psi}_{n_1,n_2}^{MLE}$ is the asymptotic normal with the mean ψ_{n_1,n_2} and asymptotic variance $\sigma^2_{\psi_{n_1,n_2}}=\sum_{j=1}^{3}\sum_{i=1}^{3}\frac{\partial\,\psi_{n_1,n_2}}{\partial\Theta_i}\frac{\partial\,\psi_{n_1,n_2}}{\partial\Theta_j}\,I_{ij}^{-1}$. Where I_{ij}^{-1} is the inverse of Fisher information matrix ,$\frac{\partial\,\psi_{n_1,n_2}}{\partial\alpha}=k_9+k_{10}+k_{11}+k_{12}$, $\frac{\partial\,\psi_{n_1,n_2}}{\partial\beta_1}=k_{13}+k_{14}+k_{15}$, $\frac{\partial\,\psi_{n_1,n_2}}{\partial\beta_2}=k_{16}+k_{17}+k_{18}$,

Where, $$k_9=-\frac{\Gamma\left[\frac{1}{\alpha}\right]\,n_2\,\beta_1^{-2-\frac{1}{\alpha}}\,\beta_2\left(\sum_{i=1}^{n_1}\sum_{j=0}^{n_2-1}\frac{(-1)^{1+i+j}\,i^{-1/\alpha}\binom{n_1}{i}\binom{n_2-1}{j}}{i\,\beta_1+(1+j)\,\beta_2}\right)}{\alpha^2\,n_1^2\left(\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2}\frac{(-1)^{i+j}\binom{n_1-1}{i}\binom{n_2}{j}}{(1+i)\,\beta_1+j\,\beta_2}\right)^2}.$$

$$k_{10}=\frac{\Gamma\left[\frac{1}{\alpha}\right]\,\text{Log}[\beta_1]\,n_2\,\beta_1^{-2-\frac{1}{\alpha}}\,\beta_2\sum_{i=1}^{n_1}\sum_{j=0}^{n_2-1}\frac{(-1)^{1+i+j}\,i^{-1/\alpha}\binom{n_1}{i}\binom{n_2-1}{j}}{i\,\beta_1+(1+j)\,\beta_2}}{\alpha^3\,n_1^2\left(\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2}\frac{(-1)^{i+j}\binom{n_1-1}{i}\binom{n_2}{j}}{(1+i)\,\beta_1+j\,\beta_2}\right)^2}.$$

$$k_{11}=-\frac{\Gamma\left[\frac{1}{\alpha}\right]\text{PolyGamma}\left[0,\frac{1}{\alpha}\right]\,n_2\,\beta_1^{-2-\frac{1}{\alpha}}\,\beta_2\sum_{i=1}^{n_1}\sum_{j=0}^{n_2-1}\frac{(-1)^{1+i+j}\,i^{-1/\alpha}\binom{n_1}{i}\binom{n_2-1}{j}}{i\,\beta_1+(1+j)\,\beta_2}}{\alpha^3\,n_1^2\left(\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2}\frac{(-1)^{i+j}\binom{n_1-1}{i}\binom{n_2}{j}}{(1+i)\,\beta_1+j\,\beta_2}\right)^2}.$$

$$k_{12}=\frac{\Gamma\left[\frac{1}{\alpha}\right]\,n_2\,\beta_1^{-2-\frac{1}{\alpha}}\,\beta_2\sum_{i=1}^{n_1}\sum_{j=0}^{n_2-1}\frac{(-1)^{1+i+j}\,i^{-1/\alpha}\binom{n_1}{i}\binom{n_2-1}{j}\,\text{Log}[i]}{\alpha^2\,(i\,\beta_1+(1+j)\,\beta_2)}}{\alpha\,n_1^2\left(\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2}\frac{(-1)^{i+j}\binom{n_1-1}{i}\binom{n_2}{j}]}{(1+i)\,\beta_1+j\,\beta_2}\right)^2}.$$

$$k_{13}=\frac{\Gamma\left[\frac{1}{\alpha}\right]\,n_2\,\beta_1^{-2-\frac{1}{\alpha}}\,\beta_2\sum_{i=1}^{n_1}\sum_{j=0}^{n_2-1}\frac{(-1)^{2+i+j}\,i^{1-\frac{1}{\alpha}}\binom{n_1}{i}\binom{n_2-1}{j}}{(i\,\beta_1+(1+j)\,\beta_2)^2}}{\alpha n_1^2\left(\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2}\frac{(-1)^{i+j}\binom{n_1-1}{i}\binom{n_2}{j}}{(1+i)\,\beta_1+j\,\beta_2}\right)^2}.$$

$$k_{14}=\frac{-\left(\sum_{i=1}^{n_1}\sum_{j=0}^{-1+n_2}\frac{(-1)^{1+i+j}\,i^{-1/\alpha}\binom{n_1}{i}\binom{n_2-1}{j}}{i\,\beta_1+(1+j)\,\beta_2}\right)}{\alpha n_1^2\left(\sum_{i=0}^{-1+n_1}\sum_{j=0}^{n_2}\frac{(-1)^{i+j}\binom{n_1-1}{i}\binom{n_2}{j}}{(1+i)\,\beta_1+j\,\beta_2}\right)^3}\;*$$

$$\left[2\Gamma\left[\frac{1}{\alpha}\right]\, n_2\, \beta_1^{-2-\frac{1}{\alpha}}\, \beta_2\left(\sum_{i=0}^{-1+n_1}\sum_{j=0}^{n_2}\frac{(-1)^{1+i+j}(1+i)\binom{n_1-1}{i}\binom{n_2}{j}}{((1+i)\,\beta_1+j\,\beta_2)^2}\right)\right].$$

$$k_{15}=\frac{\left(-2-\frac{1}{\alpha}\right)\Gamma\left[\frac{1}{\alpha}\right]\, n_2\, \beta_1^{-3-\frac{1}{\alpha}}\, \beta_2\sum_{i=1}^{n_1}\sum_{j=0}^{-1+n_2}\frac{(-1)^{1+i+j}\, i^{-1/\alpha}\binom{n_1}{i}\binom{n_2-1}{j}}{i\,\beta_1+(1+j)\,\beta_2}}{\alpha\, n_1^2\left(\sum_{i=0}^{-1+n_1}\sum_{j=0}^{n_2}\frac{(-1)^{i+j}\binom{n_1-1}{i}\binom{n_2}{j}}{(1+i)\,\beta_1+j\,\beta_2}\right)^2}.$$

$$k_{16}=\frac{\Gamma\left[\frac{1}{\alpha}\right]\, n_2\, \beta_1^{-2-\frac{1}{\alpha}}\beta_2\sum_{i=1}^{n_1}\sum_{j=0}^{n_2-1}\frac{(-1)^{2+i+j}\, i^{-1/\alpha}\,(1+j)\binom{n_1}{i}\binom{n_2-1}{j}}{(i\,\beta_1+(1+j)\,\beta_2)^2}}{\alpha\, n_1^2\left(\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2}\frac{(-1)^{i+j}\binom{n_1-1}{i}\binom{n_2}{j}}{(1+i)\,\beta_1+j\,\beta_2}\right)^2}.$$

$$k_{17}=\frac{-[2\,\Gamma\left[\frac{1}{\alpha}\right]\, n_2\, \beta_1^{-2-\frac{1}{\alpha}}\, \beta_2(\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2}\frac{(-1)^{1+i+j}\, j\binom{n_1-1}{i}\binom{n_2}{j}}{((1+i)\,\beta_1+j\,\beta_2)^2})]}{\alpha\, n_1^2\left(\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2}\frac{(-1)^{i+j}\binom{n_1-1}{i}\binom{n_2}{j}}{(1+i)\,\beta_1+j\,\beta_2}\right)^3}$$

$$*\left[\sum_{i=1}^{n_1}\sum_{j=0}^{n_2-1}\frac{(-1)^{1+i+j}\, i^{-1/\alpha}\binom{n_1}{i}\binom{n_2-1}{j}}{i\,\beta_1+(1+j)\,\beta_2}\right].$$

$$k_{18}=\frac{\Gamma\left[\frac{1}{\alpha}\right] n_2\, \beta_1^{-2-\frac{1}{\alpha}}\sum_{i=1}^{n_1}\sum_{j=0}^{n_2-1}\frac{(-1)^{1+i+j}\, i^{-1/\alpha}\binom{n_1}{i}\binom{n_2-1}{j}}{i\,\beta_1+(1+j)\,\beta_2}}{\alpha n_1^2\left(\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2}\frac{(-1)^{i+j}\binom{n_1-1}{i}\binom{n_2}{j}}{(1+i)\,\beta_1+j\,\beta_2}\right)^2},$$

Then $100(1-\alpha)\%$ A.C.I of ψ_{n_1,n_2} is given by, $\psi_{n_1,n_2} \in (\hat{\psi}^{MLE}_{n_1,n_2} \pm Z_{\frac{\alpha}{2}}\, \hat{\sigma}_{\psi_{n_1,n_2}})$,
where $Z_{\frac{\alpha}{2}}$ is the upper $\alpha/2$ th quartile of the standard normal distribution and $\hat{\sigma}_{\psi_{n_1,n_2}}$ is the value of the standard deviation at the MLE of the parameter.

3.3 Bootstrap Confidence Interval of ψ_{n_1,n_2}

(22) suggested bootstrap method Boot-p. We use Boot-p to evaluate the confidence interval of ψ_{n_1,n_2} as follows:

1. For given values of β_1, β_2 and α simulate two independent random samples $X_1,\dots,X_n$ and $Y_1,\dots,Y_m$ from $X\sim WE(\alpha,\beta_1)$ and $Y\sim WE(\alpha,\beta_2)$. respectively. To compute $\hat{\psi}^{MLE}_{n_1,n_2}$.

2. Generate independent bootstrap samples $X^*{}_1,\dots,X^*{}_n$ and $Y^*{}_1,\dots,Y^*{}_m$ using $\hat{\beta}_1$, $\hat{\beta}_2$ and $\hat{\alpha}$. To compute $\widehat{\psi^*}_{n_1,n_2} = \psi_{n_1,n_2}(\widehat{\beta^*}_1, \widehat{\beta^*}_2, \widehat{\alpha^*})$.

3. Repeat Step 2 B-times.

4. Compute the approximate $(1-\alpha)\%$ Boot-p confidence interval of Ras $(\hat{\psi}_{Boot-P}(\frac{\alpha}{2}),\hat{\psi}_{Boot-p}(1-\frac{\alpha}{2}))$ where $\alpha = 0.05$and $\hat{\psi}_{Boot-P}$ is the cumulative distribution of $\widehat{\psi^*}_{n_1,n_2}$.

3.4 Bayes Estimator for ψ_{n_1,n_2}

A Bayesian analysis is considered to incorporate prior information and to provide a probabilistic framework for inference on the unknown model parameters. In this section, the posterior distribution is generally not available in closed form because of the complexity of the likelihood function associated with the Weibull stress–strength model and the mean remaining strength of the parallel system. Therefore, two complementary computational approaches are employed. First, Lindley's approximation is used to obtain approximate Bayesian estimates of the model parameters and the reliability characteristics by approximating the required posterior expectations around the maximum likelihood estimates. Second, Markov chain

Monte Carlo (MCMC) methods are employed to numerically generate samples from the posterior distribution and obtain Bayesian estimates. Suppose that all parameters $\alpha, \beta_1, and\ \beta_2$ are unknown and have independent Gamma prior distributions with parameters (a_i, b_i), $i = 1,2,3$ respectively. Hence,

$$\text{f}(\alpha) = \frac{b_1^{a_1}}{\Gamma(a_1)}\ \alpha^{a_1-1} e^{-\alpha\, b_1}, \qquad \alpha > 0, a_1, b_1 > 0,$$
$$\text{f}(\beta_1) = \frac{b_2^{a_2}}{\Gamma(a_2)}\ \beta_1^{a_2-1}\ e^{-\beta_1\, b_2}, \qquad \beta_1 > 0, a_2, b_2 > 0,$$
$$\text{f}(\beta_2) = \frac{b_3^{a_3}}{\Gamma(a_3)}\ \beta_2^{a_3-1}\ e^{-\beta_3\, b_3}, \qquad \beta_2 > 0, a_3, b_3 > 0.$$

We can write the likelihood function of v and w as, $L \propto\ \alpha^{n+m}\ \beta_1^{n}\ \beta_2^{m} c_1 c_2$,
where,

$$c_1 = Exp[-\beta_1\ \textstyle\sum_{i=1}^{n} v_i^{\alpha} + (n_1 - 1)\sum_{i=1}^{n} Ln\left[1 - e^{-\beta_1\, v_i}\right] + (\alpha - 1)\sum_{i=1}^{n} Ln[v_i]].$$
$$c_2 = Exp[-\beta_2\ \textstyle\sum_{j=1}^{m} w_j^{\alpha} + (n_2 - 1)\sum_{j=1}^{m} Ln\left[1 - e^{-\beta_2\, w_j}\right] + (\alpha - 1)\sum_{j=1}^{m} Ln[w_j]].$$

Then, the joint posterior density of α, β_1, and β_2 is,

$$\pi(\alpha, \beta_1, \beta_2|v, w) \propto \alpha^{n+m+a_1-1}\ \beta_1^{n+a_2-1} \beta_2^{m+a_3-1}\ c_1\ c_2\ exp[-(\alpha\ b_1 + \beta_1\ b_2 + \beta_2\ b_3)].$$

Then, the Bayes estimator of ϕ_{n_1,n_2} under squared error loss function (SELF) is,

$$\hat{\psi}_{n_1,n_2}^{Bayes} = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \psi_{n_1,n_2}\ \pi(\alpha, \beta_1, \beta_2|v, w) d\alpha\ d\beta_1 d\beta_2.$$

Since we can't get the closed form of $\hat{\psi}_{Bayes}$, then we use the Lindley approximation and MCMC. Gamma prior distributions are assigned to the unknown positive parameters because of their natural support on the positive real line and their flexibility in representing different levels of prior information. The hyperparameters are selected to provide weakly informative priors centered at plausible parameter values while allowing the observed data to dominate the posterior inference. Prior independence is assumed in the absence of reliable prior information regarding dependence among the model parameters. Since the resulting posterior distribution is not available in closed form, Lindley's approximation and MCMC methods are employed to obtain Bayesian estimates. A sensitivity analysis using alternative hyperparameter values is also considered to assess the robustness of the Bayesian conclusions to the prior specification.

- **Lindley Approximation**

In this subsection we use the Lindley approximation of Bayes estimator which is introduced by (23). Lindley's method approximates the log-likelihood around the MLE using a second-order Taylor expansion is given by,

$$\hat{\psi}_{n_1,n_2}^{Bayes} = \hat{\psi}_{n_1,n_2}^{MLE} + \frac{1}{2}\sum_i\sum_j(\psi_{ij} + 2\,\psi_i\,\rho_i)\,\sigma_{ij} + \frac{1}{2}\sum_i\sum_j\sum_k\sum_l L_{ijk}\,\sigma_{ij}\,\sigma_{lk}\,\psi_l\ . \tag{5}$$

Where, $\hat{\theta} = (\hat{\alpha}, \hat{\beta}_1, \hat{\beta}_2)$ is the maximum likelihood estimator of $\theta = (\alpha, \beta_1, \beta_2)$,
$\psi_{ij} = \frac{\partial^2\,\psi}{\partial\theta_i\,\partial\theta_j}$, $\psi_i = \frac{\partial\psi}{\partial\theta_i}$, $\rho_i = \frac{\partial\rho}{\partial\theta_i}$, σ_{ij} =(i, j) element of inverse of information and $L_{ijk} = \frac{\partial^3 Ln\ L}{\partial\theta_i\,\partial\theta_j\partial\theta_k}$ all computations are evaluated at MLE of the parameters. For more details see (24). The derivatives used under Lindley approximation are shown as,

$$L_{222} = \frac{2n}{\beta_1^3} - (-1 + n_1)\sum_{i=1}^{n}\left(\frac{2e^{-3\beta_1 (x_i)^{\alpha}}{x_i}^{3\alpha}}{\left(1-e^{-\beta_1 (x_i)^{\alpha}}\right)^3} + \frac{3e^{-2\beta_1 (x_i)^{\alpha}}{x_i}^{3\alpha}}{\left(1-e^{-\beta_1 (x_i)^{\alpha}}\right)^2} + \frac{e^{-\beta_1 (x_i)^{\alpha}}{x_i}^{3\alpha}}{1-e^{-\beta_1 (x_i)^{\alpha}}}\right).$$

$$L_{333} = \frac{2m}{\beta_2^3} - (-1 + n_2)\sum_{j=1}^{m}\left(\frac{2e^{-3\beta_2 {y_j}^{\alpha}}{y_j}^{3\alpha}}{(1-e^{-\beta_2 {y_j}^{\alpha}})^3} + \frac{3e^{-2\beta_2 {y_j}^{\alpha}}{y_j}^{3\alpha}}{(1-e^{-\beta_2 {y_j}^{\alpha}})^2} + \frac{e^{-\beta_2 {y_j}^{\alpha}}{y_j}^{3\alpha}}{1-e^{-\beta_2 {y_j}^{\alpha}}}\right).$$

$$\begin{aligned} L_{111} = {} & \frac{2(m+n)}{\alpha^3} - \beta_1\ \textstyle\sum_{i=1}^{n}\text{Log}[x_i]^3 {x_i}^{\alpha} - (-1 + n_1)\displaystyle\sum_{i=1}^{n}\Bigg(\frac{e^{-\beta_1 {x_i}^{\alpha}}\text{Log}[x_i]^3\beta_1 {x_i}^{\alpha}}{1-e^{-\beta_1 {x_i}^{\alpha}}} - \frac{3e^{-2\beta_1 {x_i}^{\alpha}}\text{Log}[x_i]^3\beta_1^2 {x_i}^{2\alpha}}{(1-e^{-\beta_1 {x_i}^{\alpha}})^2} - \\ & \frac{3e^{-\beta_1 {x_i}^{\alpha}}\text{Log}[x_i]^3\beta_1^2 {x_i}^{2\alpha}}{1-e^{-\beta_1 {x_i}^{\alpha}}} + \frac{2e^{-3\beta_1 x[i]^{\alpha}}\text{Log}[x_i]^3\beta_1^3 {x_i}^{3\alpha}}{(1-e^{-\beta_1 {x_i}^{\alpha}})^3} + \frac{3e^{-2\beta_1 {x_i}^{\alpha}}\text{Log}[x_i]^3\beta_1^3 x[i]^{3\alpha}}{(1-e^{-\beta_1 {x_i}^{\alpha}})^2} + \frac{e^{-\beta_1 {x_i}^{\alpha}}\text{Log}[x_i]^3\beta_1^3 {x_i}^{3\alpha}}{1-e^{-\beta_1 x_i}}\Bigg) - \\ & \beta_2\ \textstyle\sum_{j=1}^{m}\text{Log}[y[j]]^3 y[j]^{\alpha} - (-1 + n_2)\displaystyle\sum_{j=1}^{m}\Bigg(\frac{e^{-\beta_2 {y_j}^{\alpha}}\text{Log}[y_j]^3\beta_2 {y_j}^{\alpha}}{1-e^{-\beta_2 {y_j}^{\alpha}}} - \frac{3e^{-2\beta_2 {y_j}^{\alpha}}\text{Log}[y_j]^3\beta_2^2 {y_j}^{2\alpha}}{(1-e^{-\beta_2 {y_j}^{\alpha}})^2} - \frac{3e^{-\beta_2 {y_j}^{\alpha}}\text{Log}[y_j]^3\beta_2^2 {y_j}^{2\alpha}}{1-e^{-\beta_2 {y_j}^{\alpha}}} + \\ & \frac{2e^{-3\beta_2 {y_j}^{\alpha}}\text{Log}[y_j]^3\beta_2^3 {y_j}^{3\alpha}}{(1-e^{-\beta_2 {y_j}^{\alpha}})^3} + \frac{3e^{-2\beta_2 {y_j}^{\alpha}}\text{Log}[y_j]^3\beta_2^3 {y_j}^{3\alpha}}{(1-e^{-\beta_2 {y_j}^{\alpha}})^2} + \frac{e^{-\beta_2 y[j]^{\alpha}}\text{Log}[y_j]^3\beta_2^3 {y_j}^{3\alpha}}{1-e^{-\beta_2 {y_j}^{\alpha}}}\Bigg). \end{aligned}$$

$$L_{112}=L_{211}=L_{121}=-\sum_{i=1}^{n}\mathrm{Log}[x_i]^2{x_i}^{\alpha}-(-1+n_1)\sum_{i=1}^{n}\Big(\frac{e^{-\beta_1 x[i]^{\alpha}}\mathrm{Log}[x_i]^2{x_i}^{\alpha}}{1-e^{-\beta_1 x[i]^{\alpha}}}+$$
$$2\mathrm{Log}[x_i]{x_i}^{\alpha}\Big(-\frac{e^{-2\beta_1 {x_i}^{\alpha}}\mathrm{Log}[x_i]\beta_1{x_i}^{\alpha}}{(1-e^{-\beta_1 x[i]^{\alpha}})^2}-\frac{e^{-\beta_1 {x_i}^{\alpha}}\mathrm{Log}[x_i]\beta_1{x_i}^{\alpha}}{1-e^{-\beta_1 {x_i}^{\alpha}}}\Big)+{x_i}^{\alpha}\Big(\frac{2e^{-2\beta_1 {x_i}^{\alpha}}\mathrm{Log}[x_i]^2\beta_1^2{x_i}^{2\alpha}}{(1-e^{-\beta_1 {x_i}^{\alpha}})^2}+$$
$$\frac{-e^{-\beta_1 {x_i}^{\alpha}}\mathrm{Log}[x_i]^2\beta_1{x_i}^{\alpha}+e^{-\beta_1 {x_i}^{\alpha}}\mathrm{Log}[x_i]^2\beta_1^2{x_i}^{2\alpha}}{1-e^{-\beta_1 {x_i}^{\alpha}}}+e^{-\beta_1 {x_i}^{\alpha}}\Big(-\frac{e^{-\beta_1 x[ix_i]^{\alpha}}\mathrm{Log}[x_i]^2\beta_1{x_i}^{\alpha}}{(1-e^{-\beta_1 {x_i}^{\alpha}})^2}+\frac{2e^{-2\beta_1 {x_i}^{\alpha}}\mathrm{Log}[x_i]^2\beta_1^2{x_i}^{2\alpha}}{(1-e^{-\beta_1 {x_i}^{\alpha}})^3}+$$
$$\frac{e^{-\beta_1 x[i]^{\alpha}}\mathrm{Log}[x[i]]^2\beta_1^2 x[i]^{2\alpha}}{(1-e^{-\beta_1 {x_i}^{\alpha}})^2}\Big)\Big)\Big).$$

$$L_{113}=L_{131}=L_{311}=-\sum_{j=1}^{m}\mathrm{Log}[y_j]^2{y_j}^{\alpha}-(-1+n_2)\sum_{j=1}^{m}\Big(\frac{e^{-\beta_2 {y_j}^{\alpha}}\mathrm{Log}[y_j]^2{y_j}^{\alpha}}{1-e^{-\beta_2 {y_j}^{\alpha}}}+$$
$$2\mathrm{Log}[y_j]{y_j}^{\alpha}\Big(-\frac{e^{-2\beta_2 {y_j}^{\alpha}}\mathrm{Log}[y_j]\beta_2{y_j}^{\alpha}}{(1-e^{-\beta_2 {y_j}^{\alpha}})^2}-\frac{e^{-\beta_2 {y_j}^{\alpha}}\mathrm{Log}[y_j]\beta_2{y_j}^{\alpha}}{1-e^{-\beta_2 {y_j}^{\alpha}}}\Big)+{y_j}^{\alpha}\Big(\frac{2e^{-2\beta_2 {y_j}^{\alpha}}\mathrm{Log}[y_j]^2\beta_2^2{y_j}^{2\alpha}}{(1-e^{-\beta_2 {y_j}^{\alpha}})^2}+$$
$$\frac{-e^{-\beta_2 {y_j}^{\alpha}}\mathrm{Log}[y_j]^2\beta_2{y_j}^{\alpha}+e^{-\beta_2 {y_j}^{\alpha}}\mathrm{Log}[y_j]^2\beta_2^2{y_j}^{2\alpha}}{1-e^{-\beta_2 {y_j}^{\alpha}}}+e^{-\beta_2 {y_j}^{\alpha}}\Big(-\frac{e^{-\beta_2 {y_j}^{\alpha}}\mathrm{Log}[y_j]^2\beta_2{y_j}^{\alpha}}{(1-e^{-\beta_2 {y_j}^{\alpha}})^2}+\frac{2e^{-2\beta_2 {y_j}^{\alpha}}\mathrm{Log}[y_j]^2\beta_2^2{y_j}^{2\alpha}}{(1-e^{-\beta_2 {y_j}^{\alpha}})^3}+$$
$$\frac{e^{-\beta_2 {y_j}^{\alpha}}\mathrm{Log}[y_j]^2\beta_2^2{y_j}^{2\alpha}}{(1-e^{-\beta_2 {y_j}^{\alpha}})^2}\Big)\Big)\Big).$$

$$L_{221}=L_{122}=-(-1+n_1)\sum_{i=1}^{n}\Big(-\frac{2e^{-2\beta_1 {x_i}^{\alpha}}\mathrm{Log}[x_i]{x_i}^{2\alpha}}{(1-e^{-\beta_1 {x_i}^{\alpha}})^2}-\frac{2e^{-\beta_1 {x_i}^{\alpha}}\mathrm{Log}[x_i]{x_i}^{2\alpha}}{1-e^{-\beta_1 {x_i}^{\alpha}}}+\frac{2e^{-3\beta_1 {x_i}^{\alpha}}\mathrm{Log}[x_i]\beta_1{x_i}^{3\alpha}}{(1-e^{-\beta_1 {x_i}^{\alpha}})^3}+$$
$$\frac{3e^{-2\beta_1 {x_i}^{\alpha}}\mathrm{Log}[x_i]\beta_1{x_i}^{3\alpha}}{(1-e^{-\beta_1 {x_i}^{\alpha}})^2}+\frac{e^{-\beta_1 {x_i}^{\alpha}}\mathrm{Log}[x_i]\beta_1{x_i}^{3\alpha}}{1-e^{-\beta_1 {x_i}^{\alpha}}}\Big),$$

$$L_{331}=L_{133}=-(-1+n_2)\sum_{j=1}^{m}\Big(-\frac{2e^{-2\beta_2 {y_j}^{\alpha}}\mathrm{Log}[y_j]{y_j}^{2\alpha}}{(1-e^{-\beta_2 {y_j}^{\alpha}})^2}-\frac{2e^{-\beta_2 {y_j}^{\alpha}}\mathrm{Log}[y_j]{y_j}^{2\alpha}}{1-e^{-\beta_2 {y_j}^{\alpha}}}+\frac{2e^{-3\beta_2 {y_j}^{\alpha}}\mathrm{Log}[y_j]\beta_2{y_j}^{3\alpha}}{(1-e^{-\beta_2 {y_j}^{\alpha}})^3}+$$
$$\frac{3e^{-2\beta_2 {y_j}^{\alpha}}\mathrm{Log}[y_j]\beta_2{y_j}^{3\alpha}}{(1-e^{-\beta_2 {y_j}^{\alpha}})^2}+\frac{e^{-\beta_2 y[j]^{\alpha}}\mathrm{Log}[y_j]\beta_2{y_j}^{3\alpha}}{1-e^{-\beta_2 {y_j}^{\alpha}}}\Big).$$

$$L_{321}=L_{132}=L_{312}=L_{213}=0.$$

$$\phi 1=\frac{RQ1-QR1}{R^2},\ \phi 2=\frac{RQ2-QR2}{R^2},\ \phi 3=\frac{RQ3-QR3}{R^2},\ \phi 11=\frac{Q11R^2-QRR11-2RR1Q1+2Q(R1)^2}{R^3},$$
$$\phi 22=\frac{Q22R^2-QRR22-2RR2Q2+2Q(R2)^2}{R^3},\ \phi 33=\frac{Q33R^2-QRR33-2RR3Q3+2Q(R3)^2}{R^3},$$
$$\phi 12=\phi 21=\frac{Q12R^2-Q1RR2-2RR1Q2+2QRR1R2}{R^3},$$
$$\phi 13=\phi 31=\frac{Q13R^2-Q1RR3-2RR1Q3+2QRR1R3}{R^3},$$
$$\phi 23=\phi 23=\frac{Q23R^2-Q2RR3-2RR2Q3+2QRR2R3}{R^3},$$
$$R1=R11=0;R12=R21=0;R13=R31=0,$$
$$R=n_1\beta_1\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2}\frac{(-1)^{i+j}\binom{n_1-1}{i}\binom{n_2}{j}}{(i+1)\beta_1+j\beta_2},$$

$$Q=\frac{n_2\beta_2\Gamma[\frac{1}{\alpha}]}{\beta_1^{\frac{1}{\alpha}}\alpha}\sum_{i=1}^{n_1}\sum_{j=0}^{n_2-1}\frac{(-1)^{i+j+1}\binom{n_1}{i}\binom{n_2-1}{j}}{(i\beta_1+(j+1)\beta_2)i^{\frac{1}{\alpha}}}.$$

$$R2=n_1\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2}(-1)^{i+j}\binom{n_1-1}{i}\binom{n_2}{j}\Big(-\frac{(1+i)\beta_1}{((1+i)\beta_1+j\beta_2)^2}+\frac{1}{(1+i)\beta_1+j\beta_2}\Big).$$

$$R22=n_1\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2}(-1)^{i+j}\binom{n_1-1}{i}\binom{n_2}{j}\Big(\frac{2(1+i)^2\beta_1}{((1+i)\beta_1+j\beta_2)^3}-\frac{2(1+i)}{((1+i)\beta_1+j\beta_2)^2}\Big).$$

$$R3 = n_1 \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2} (-1)^{i+j} \binom{n_1-1}{i} \binom{n_2}{j} \left(-\frac{j\beta_1}{((1+i)\beta_1 + j\beta_2)^2}\right).$$

$$R33 = n_1 \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2} (-1)^{i+j} \binom{n_1-1}{i} \binom{n_2}{j} \left(\frac{2j^2\beta_1}{((1+i)\beta_1 + j\beta_2)^3}\right).$$

$$R23 = R32 = n_1 \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2} (-1)^{i+j} \binom{n_1-1}{i} \binom{n_2}{j} \left(\frac{2(1+i)j\beta_1}{((1+i)\beta_1 + j\beta_2)^3} - \frac{j}{((1+i)\beta_1 + j\beta_2)^2}\right).$$

$$Q1 = n_2 \sum_{i=1}^{n_1} \sum_{j=0}^{n_2-1} \binom{n_1-1}{i} \binom{n_2}{j} \left(-\frac{i^{-1/\alpha}\Gamma\left[\frac{1}{\alpha}\right]\beta_1^{-1/\alpha}\beta_2}{\alpha^2(i\beta_1+(1+j)\beta_2)} + \frac{i^{-1/\alpha}\Gamma\left[\frac{1}{\alpha}\right]\mathrm{Log}[i]\beta_1^{-1/\alpha}\beta_2}{\alpha^3(i\beta_1+(1+j)\beta_2)} + \frac{i^{-1/\alpha}\Gamma[\frac{1}{\alpha}]\mathrm{Log}[\beta_1]\beta_1^{-1/\alpha}\beta_2}{\alpha^3(i\beta_1+(1+j)\beta_2)} - \frac{i^{-1/\alpha}\Gamma[\frac{1}{\alpha}]\mathrm{PolyGamma}[0,\frac{1}{\alpha}]\beta_1^{-1/\alpha}\beta_2}{\alpha^3(i\beta_1+(1+j)\beta_2)}\right).$$

$$Q11 = n_2 \binom{n_1}{i} \binom{n_{2.}-1}{j} \Bigg(\frac{2\mathrm{Log}[\beta_1]\beta_1^{-1/\alpha}\beta_2}{\alpha^2(i\beta_1+(1+j)\beta_2)}\Bigg(-\frac{i^{-1/\alpha}\Gamma\frac{1}{\alpha}]}{\alpha^2} + \frac{i^{-1/\alpha}\Gamma[\frac{1}{\alpha}]\mathrm{Log}[i]}{\alpha^3} - \frac{i^{-1/\alpha}\Gamma[\frac{1}{\alpha}]\mathrm{PolyGamma}[0,\frac{1}{\alpha}]}{\alpha^3}\Bigg)$$

$$+ \frac{1}{i\beta_1+(1+j)\beta_2}\Bigg(\Gamma[\frac{1}{\alpha}]\Bigg(\frac{2i^{-1/\alpha}}{\alpha^3} - \frac{2i^{-1/\alpha}\mathrm{Log}[i]}{\alpha^4} + \frac{-\frac{2i^{-1/\alpha}\mathrm{Log}[i]}{\alpha^3} + \frac{i^{-1/\alpha}\mathrm{Log}[i]^2}{\alpha^4}}{\alpha}\Bigg) - \frac{2\Gamma\left[\frac{1}{\alpha}\right]\left(-\frac{i^{-1/\alpha}}{\alpha^2} + \frac{i^{-1/\alpha}\mathrm{Log}[i]}{\alpha^3}\right)\mathrm{PolyGamma}\left[0,\frac{1}{\alpha}\right]}{\alpha^2} +$$

$$\frac{1}{\alpha} i^{-1/\alpha}\Bigg(\frac{2\Gamma\left[\frac{1}{\alpha}\right]\mathrm{PolyGamma}\left[0,\frac{1}{\alpha}\right]}{\alpha^3} + \frac{\Gamma\left[\frac{1}{\alpha}\right]\mathrm{PolyGamma}\left[0,\frac{1}{\alpha}\right]^2}{\alpha^4} + \frac{\Gamma\left[\frac{1}{\alpha}\right]\mathrm{PolyGamma}\left[1,\frac{1}{\alpha}\right]}{\alpha^4}\Bigg)\Bigg)\beta_1^{-1/\alpha}\beta_2 + \frac{i^{-1/\alpha}\Gamma[\frac{1}{\alpha}]\left(-\frac{2\mathrm{Log}[\beta_1]\beta_1^{-1/\alpha}}{\alpha^3} + \frac{\mathrm{Log}[\beta_1]^2\beta_1^{-1/\alpha}}{\alpha^4}\right)\beta_2}{\alpha(i\beta_1+(1+j)\beta_2)}\Bigg).$$

$$Q2 = n_2 \sum_{i=1}^{n_1} \sum_{j=0}^{n_2-1} (-1)^{i+j+1} \binom{n_1}{i} \binom{n_{2-1}}{j} \left(-\frac{i^{1-\frac{1}{\alpha}}\Gamma[\frac{1}{\alpha}]\beta_1^{-1/\alpha}\beta_2}{\alpha(i\beta_1+(1+j)\beta_2)^2} - \frac{i^{-1/\alpha}\Gamma[\frac{1}{\alpha}]\beta_1^{-1-\frac{1}{\alpha}}\beta_2}{\alpha^2(i\beta_1+(1+j)\beta_2)}\right).$$

$$Q22 = n_2 \sum_{i=1}^{n_1} \sum_{j=0}^{n_2-1} (-1)^{i+j+1} \binom{n_1}{i} \binom{n_2-1}{j} \left(\frac{i^{-1/\alpha}\Gamma[\frac{1}{\alpha}]\left(\frac{2i^2\beta_1^{-1/\alpha}\beta_2}{(i\beta_1+(1+j)\beta_2)^3} + \frac{2i\beta_1^{-1-\frac{1}{\alpha}}\beta_2}{\alpha(i\beta_1+(1+j)\beta_2)^2} - \frac{(-1-\frac{1}{\alpha})\beta_1^{-2-\frac{1}{\alpha}}\beta_2}{\alpha(i\beta_1+(1+j)\beta_2)}\right)}{\alpha}\right).$$

$$Q3 = n_2 \sum_{i=1}^{n_1} \sum_{j=0}^{n_2-1} (-1)^{i+j+1} \binom{n_1}{i} \binom{n_2}{j} - 1\Bigg)\left(-\frac{i^{-1/\alpha}(1+j)\Gamma\left[\frac{1}{\alpha}\right]\beta_1^{-1/\alpha}\beta_2}{\alpha(i\beta_1+(1+j)\beta_2)^2} + \frac{i^{-1/\alpha}\Gamma[\frac{1}{\alpha}]\beta_1^{-1/\alpha}}{\alpha(i\beta_1+(1+j)\beta_2)}\right).$$

$$Q33 = n_2 \sum_{i=1}^{n_1} \sum_{j=0}^{n_2-1} (-1)^{i+j+1} \binom{n_1}{i} \binom{n_2}{j} - 1\Bigg)\left(\frac{i^{-1/\alpha}\Gamma[\frac{1}{\alpha}]\left(\frac{2(1+j)^2\beta_1^{-1/\alpha}\beta_2}{(i\beta_1+(1+j)\beta_2)^3} - \frac{2(1+j)\beta_1^{-1/\alpha}}{(i\beta_1+(1+j)\beta_2)^2}\right)}{\alpha}\right).$$

$$Q21 = Q12 = n_2 \sum_{i=1}^{n_1} \sum_{j=0}^{n_2-1} (-1)^{i+j+1} \binom{n_1}{i} \binom{n_2}{j} - 1\Bigg)\left(\frac{i^{1-\frac{1}{\alpha}}\Gamma\left[\frac{1}{\alpha}\right]\beta_1^{-1/\alpha}\beta_2}{\alpha^2(i\beta_1+(1+j)\beta_2)^2}\right.$$

$$-\frac{i^{1-\frac{1}{\alpha}}\Gamma\left[\frac{1}{\alpha}\right]\mathrm{Log}[i]\beta_1^{-1/\alpha}\beta_2}{\alpha^3(i\beta_1+(1+j)\beta_2)^2}-\frac{i^{1-\frac{1}{\alpha}}\Gamma\left[\frac{1}{\alpha}\right]\mathrm{Log}[\beta_1]\beta_1^{-1/\alpha}\beta_2}{\alpha^3(i\beta_1+(1+j)\beta_2)^2}$$
$$+\frac{i^{1-\frac{1}{\alpha}}\Gamma\left[\frac{1}{\alpha}\right]\mathrm{PolyGamma}\left[0,\frac{1}{\alpha}\right]\beta_1^{-1/\alpha}\beta_2}{\alpha^3(i\beta_1+(1+j)\beta_2)^2}+\frac{2i^{-1/\alpha}\Gamma\left[\frac{1}{\alpha}\right]\beta_1^{-1-\frac{1}{\alpha}}\beta_2}{\alpha^3(i\beta_1+(1+j)\beta_2)}$$
$$-\frac{i^{-1/\alpha}\Gamma\left[\frac{1}{\alpha}\right]\mathrm{Log}[i]\beta_1^{-1-\frac{1}{\alpha}}\beta_2}{\alpha^4(i\beta_1+(1+j)\beta_2)}-\frac{i^{-1/\alpha}\Gamma[\frac{1}{\alpha}]\mathrm{Log}[\beta_1]\beta_1^{-1-\frac{1}{\alpha}}\beta_2}{\alpha^4(i\beta_1+(1+j)\beta_2)}+\frac{i^{-1/\alpha}\Gamma[\frac{1}{\alpha}]\mathrm{PolyGamma}[0,\frac{1}{\alpha}]\beta_1^{-1-\frac{1}{\alpha}}\beta_2}{\alpha^4(i\beta_1+(1+j)\beta_2)}).$$

$$\mathrm{Q23}=\mathrm{Q32}=n_2\sum_{i=1}^{n_1}\sum_{j=0}^{n_2-1}(-1)^{i+j+1}\binom{n_1}{i}\left(\binom{n_2}{j}-1\right)(\frac{2i^{1-\frac{1}{\alpha}}(1+j)\Gamma\left[\frac{1}{\alpha}\right]\beta_1^{-1/\alpha}\beta_2}{\alpha(i\beta_1+(1+j)\beta_2)^3}-\frac{i^{1-\frac{1}{\alpha}}\Gamma\left[\frac{1}{\alpha}\right]\beta_1^{-1/\alpha}}{\alpha(i\beta_1+(1+j)\beta_2)^2}+$$
$$\frac{i^{-1/\alpha}(1+j)\Gamma[\frac{1}{\alpha}]\beta_1^{-1-\frac{1}{\alpha}}\beta_2}{\alpha^2(i\beta_1+(1+j)\beta_2)^2}-\frac{i^{-1/\alpha}\Gamma[\frac{1}{\alpha}]\beta_1^{-1-\frac{1}{\alpha}}}{\alpha^2(i\beta_1+(1+j)\beta_2)}).$$

$$\mathrm{Q13}=\mathrm{Q31}=n_2\sum_{i=1}^{n_1}\sum_{j=0}^{n_2-1}(-1)^{i+j+1}\binom{n_1}{i}\left(\binom{n_2}{j}-1\right)(\frac{i^{-1/\alpha}(1+j)\Gamma\left[\frac{1}{\alpha}\right]\beta_1^{-1/\alpha}\beta_2}{\alpha^2(i\beta_1+(1+j)\beta_2)^2}-$$
$$\frac{i^{-1/\alpha}(1+j)\Gamma\left[\frac{1}{\alpha}\right]\mathrm{Log}[i]\beta_1^{-1/\alpha}\beta_2}{\alpha^3(i\beta_1+(1+j)\beta_2)^2}-\frac{i^{-1/\alpha}(1+j)\Gamma\left[\frac{1}{\alpha}\right]\mathrm{Log}[\beta_1]\beta_1^{-1/\alpha}\beta_2}{\alpha^3(i\beta_1+(1+j)\beta_2)^2}$$
$$+\frac{i^{-1/\alpha}(1+j)\Gamma\left[\frac{1}{\alpha}\right]\mathrm{PolyGamma}\left[0,\frac{1}{\alpha}\right]\beta_1^{-1/\alpha}\beta_2}{\alpha^3(i\beta_1+(1+j)\beta_2)^2}-$$
$$\frac{i^{-1/\alpha}\Gamma\left[\frac{1}{\alpha}\right]\beta_1^{-1/\alpha}}{\alpha^2(i\beta_1+(1+j)\beta_2)}+\frac{i^{-1/\alpha}\Gamma[\frac{1}{\alpha}]\mathrm{Log}[i]\beta_1^{-1/\alpha}}{\alpha^3(i\beta_1+(1+j)\beta_2)}+\frac{i^{-1/\alpha}\Gamma[\frac{1}{\alpha}]\mathrm{Log}[\beta_1]\beta_1^{-1/\alpha}}{\alpha^3(i\beta_1+(1+j)\beta_2)}-\frac{i^{-1/\alpha}\Gamma[\frac{1}{\alpha}]\mathrm{PolyGamma}[0,\frac{1}{\alpha}]\beta_1^{-1/\alpha}}{\alpha^3(i\beta_1+(1+j)\beta_2)}).$$

$\rho 1=\frac{\mathrm{a1}-1}{\alpha}-\mathrm{b1};\ \rho 2=\frac{\mathrm{a2}-1}{\beta_1}-\mathrm{b2};\ \rho 3=\frac{\mathrm{a3}-1}{\beta_2}-\mathrm{b3}.$

- **Monte-Carlo Markov Chain (MCMC)**

The marginal posterior density function of α, β_1, β_2 are given respectively,

$$\pi(\beta_1|\alpha,\beta_2,v,w)\propto\ \beta_1^{n+a_2-1}\ c_1,\qquad \pi(\beta_2|\alpha,\beta_1,v,w)\propto\ \beta_2^{m+a_3-1}\ c_2,$$
$$\pi(\alpha|\beta_1,\beta_2,v,w)\propto\alpha^{n+m+a_1-1}\ exp[-(\alpha\, b_1+\beta_1\, b_2+\beta_2\, b_3)].$$

Now to get the Bayesian estimator, we use hybrid Metropolis- Hasting and Gibbs sampling algorithm see (25) as follows:

1. Choose initial values $\alpha^{(0)}, \beta_1^{(0)}$ and $\beta_2^{(0)}$.

2. Generate $\alpha^{(k)}, \beta_1^{(k)}$ and $\beta_2^{(k)}$ from marginal posterior.

3. Compute $\hat{\psi}_{n_1,n_2}^{Bayes\ (k)}$ at $\alpha^{(k)}, \beta_1^{(k)}$ and $\beta_2^{(k)}$.

4. Repeat step 2 and 3 N-times and compute,

$$\hat{\psi}_{n_1,n_2}^{Bayes\ MCMC}=\frac{1}{N-M}\sum_{k=M+1}^{N-M}\hat{\psi}_{n_1,n_2}^{Bayes\ (k)},\tag{6}$$

where M is the burn-in period.

Figure (3) explain MCMC algorithm as follows

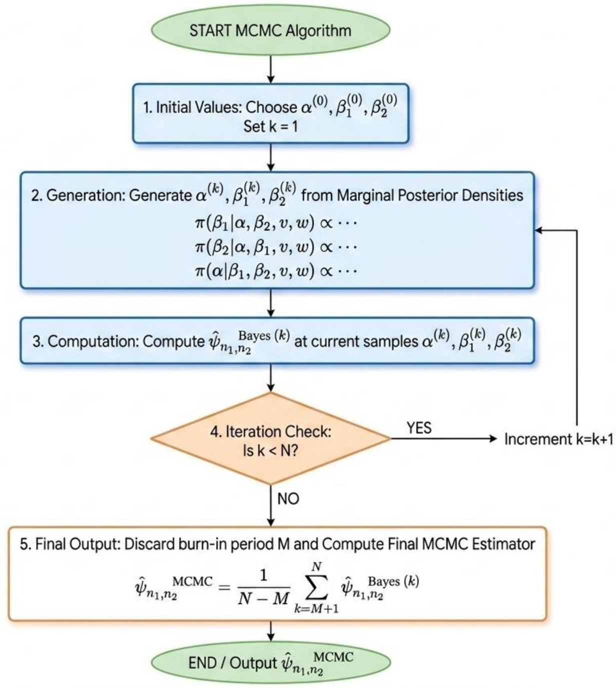

Figure 3: Flowchart of MCMC Algorithm

- **Hyperparameters Sensitivity Analysis for Weibull Distribution**

Sensitivity analysis is very important to determine the effect of hyperparameters on the posterior. To determine the sensitivity of hyperparameters for Weibull distribution, we consider three schemes:

Scheme I: $(a_1, b_1) = (16,1), (a_2, b_2) = (2,1),\ (a_3, b_3) = (3,1)$.

Scheme II: $(a_1, b_1) = (1,1), (a_2, b_2) = (2,2),\ (a_3, b_3) = (3,3)$.

Scheme III: $(a_1, b_1) = (1,2), (a_2, b_2) = (2,5),\ (a_3, b_3) = (1,3)$.

Table 1 shows the results of sensitivity analysis; we see no big difference between the three schemes. The small differences among the posterior summaries can be attributed to the fact that the likelihood function contains sufficient information from the observed data to dominate the prior contribution. Consequently, although the prior schemes differ, their influence on the resulting posterior distributions is limited. This indicates that the proposed Bayesian estimation procedure is not sensitive to reasonable changes in the prior specification.

4. Simulation Study

We perform Monte-Carlo simulation study to compare the performance of two-point estimators of ψ_{n_1,n_2}. In this comparison, we use bias and (MSE) and the relative efficiency (RE). when $\hat{\theta}$ is an estimator of a parameter θ, the empirical bias of $\hat{\theta}$ is given by,

$$B(\hat{\theta}, \theta) = \frac{1}{N} \sum_{i=1}^{N} (\hat{\theta}_i - \theta),$$

The empirical mean square error of $\hat{\theta}$ is given by,

$$MSE(\hat{\theta}, \theta) = \frac{1}{N} \sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2,$$

The empirical relative efficiency between $\hat{\theta}_1$and $\hat{\theta}_2$ is given by,

$$RE(\hat{\theta}_1, \hat{\theta}_2, \theta) = \frac{MSE(\hat{\theta}_1, \theta)}{MSE(\hat{\theta}_2, \theta)}.$$

The Monte-Carlo study compares between 99% (A.C.I.) and 99% (B.C.I.) using in this comparison average confidence length (A.C.L.) and converge probabilities (CP). All computations in this simulation study are performed using Mathematica 11 and R- Program according to the following algorithm. All computations were executed using Mathematica 11 and R. Each simulation run consisted of $N = 10{,}000$ Monte Carlo iterations, with 5,000 burn-in samples utilized for MCMC draws:

i. Generate random samples for $(n, m) = (10,10), (20,20), (30,30), (40,40)\, and\ (50,50)$

ii. Use the initial value of parameters $(\alpha, \beta_1, \beta_2) = (16, 2.0056, 3.0614)$ because we use the prior of these parameters $(a_1, b_1) = (16,1)$, $(a_2, b_2) = (2,1)$, and $(a_3, b_3) = (3,1)$.

iii. Calculate the true value of ψ_{n_1,n_2} at initial value of parameters and $n_1 = n_2 = 5$ and $n_1 = n_2 = 10$.

iv. Calculate $\hat{\psi}_{n_1,n_2}^{MLE}$, $\hat{\psi}_{n_1,n_2}^{Bayes}$ and $\hat{\psi}_{n_1,n_2}^{Bayes\ MCMC}$using equations (4), (5) and (6) respectively.

v. Calculate bias, MSE and RE.

vi. Calculate ACL and CP for 99% ACI and 99% BCI

The results of point estimators and interval estimators are shown in Table 2 and Table 3 respectively. From Table 2, since bias, MSE for MLE is less than its corresponding for Bayes estimator and RE1 < 1, then we can conclude that the $\hat{\psi}^{MLE}_{n_1,n_2}$ is more efficient than $\hat{\psi}^{Bayes}_{n_1,n_2}$. RE2 < 1, then we can conclude that the $\hat{\psi}^{MLE}_{n_1,n_2}$ is more efficient than $\hat{\psi}^{Bayes\ MCMC}_{n_1,n_2}$. RE3 < 1, then we can conclude that the $\hat{\psi}^{Bayes}_{n_1,n_2}$ is more efficient than $\hat{\psi}^{Bayes\ MCMC}_{n_1,n_2}$. Table 3 shows that the ACL for BCI is shorter than the corresponding length for ACI for all sample sizes, finally the converge probabilities for both ACI and BCI are close to the nominal 99%.

5. Real Example: Breast Cancer Disease

To demonstrate the practical applicability of the proposed estimation techniques within a stress-strength paradigm, we examine two empirical datasets detailing breast cancer intervention outcomes. Evaluating the comparative performance of monotherapy versus combined therapeutic regimens is fundamental in clinical oncology to optimize efficacy while mitigating adverse effects.

Here, the random variable X denotes the number of tumors managed via chemotherapy alone (representing system strength), whereas Y corresponds to the tumor count treated through combined chemotherapy and radiotherapy (representing stress). Under this operationalization, the stress-strength reliability metric $R = P(Y < X)$ reflects the probability that single-agent chemotherapy yields a superior clinical response compared to the dual-treatment protocol.

Data set I (X: Number of tumours treated in breast cancer disease with chemotherapy):

496; 532; 475; 294; 552; 254; 154; 1941; 2029; 129; 117; 74; 523; 472; 480; 461; 413; 20; 2329; 122; 313; 191; 1274; 449; 731; 768; 850.

Data set II (Y: Number of tumours treated in breast cancer disease with combined with chemotherapy and radiotherapy):

479; 420; 282; 99; 569; 191; 102; 1556; 1704; 87; 34; 24; 436; 365; 351; 426; 271; 10; 1839; 143; 439; 203;727; 337; 630; 617; 602.

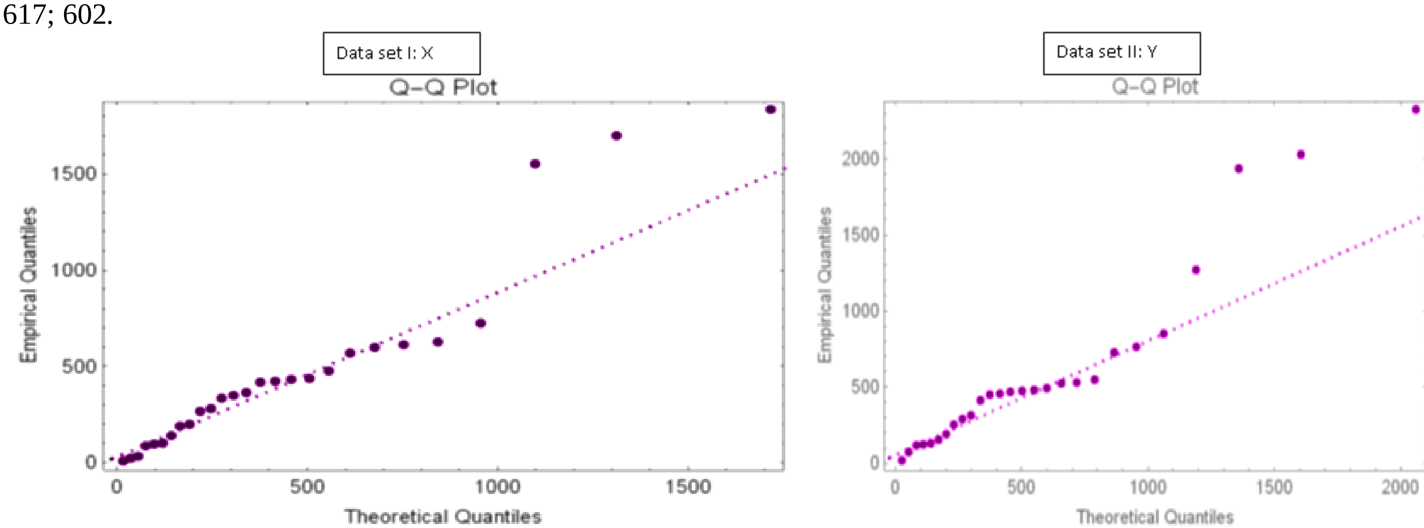

Figure 4: Q-Q Plot for data X and data Y

First: The Q-Q plot is based on the maximum likelihood estimates of α , β_1, and β_2 to show that the Weibull distribution is a good fit for the two data sets of number of tumours treated in breast cancer disease see Figure (4). To complement the visual evidence provided by quantile-quantile (Q-Q) plots, we evaluated distributional concordance through three formal goodness-of-fit hypothesis tests: Kolmogorov–Smirnov (D) statistic, Anderson–Darling (A^2) statistic, Cramér–von Mises (W^2) statistic. The high p-values associated with all three testing procedures ($p > 0.05$) indicate that the null hypothesis of Weibull distributional conformity cannot be rejected for either dataset. For results see Table 4 and Table 5. Framing treatment evaluation as a stress-strength problem provides clinicians with a quantitative tool to assess whether adding radiotherapy provides sufficient marginal benefit to justify the additional physiological burden.

6. Conclusions

In this paper, we examined the mean remaining strength (MRS) of a component X subjected to stress Y, where both strength and stress follow Weibull distributions. We derived the conditional distribution of the remaining strength in a parallel system exposed to parallel stress. Additionally, we explored the likelihood ratio ordering for two-component systems. Both maximum likelihood and Bayesian estimators of the MRS were obtained. Based on relative efficiency (RE) calculations, the maximum likelihood estimator was found to outperform the Bayesian estimator in terms of efficiency. Regarding interval estimation, our results indicate that the bootstrap confidence interval (BCI) provides better performance compared to the asymptotic confidence interval (ACI). Finally, we demonstrated the practical use of the proposed estimators using a dataset related to breast cancer treatment. Despite these contributions, the proposed methodology has several limitations that highlight natural avenues for future extension. The current formulation assumes equal Weibull shape parameters for stress and strength variables. Future work will extend this framework to accommodate unequal shape parameters which introduce additional parameter complexity but broadens practical applicability. independence assumption**:** X and Y were modeled as independent random variables. In complex physical and biological systems, stress and strength often exhibit mutual dependence. Future research could incorporate copula-based dependency structures to model dependent stress–strength interactions.

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Annexures

Table 1: *Prior Sensitivity Analysis for Weibull Distribution*

Prior	Parameters	Mean	Median	Variance	95% CI
Scheme I	α	1.0002	0.6925	1.0028	(0.0506,2.9963)
	β_1	0.9977	0.6919	0.9953	(0.0506,2.9908)
	β_2	1.0039	0.6964	1.0048	(0.0520,3.0038)
Scheme II	α	0.9992	0.6942	0.9992	(0.0518,2.9898)
	β_1	1.0013	0.6904	1.0048	(0.0513,3.0107)
	β_2	0.9947	0.6882	0.9860	(0.0512,2.9845)
Scheme III	α	0.9947	0.6874	0.9982	(0.0507,3.0034)
	β_1	0.9999	0.6904	0.10059	(0.0513,3.0045)
	β_2	1.0002	0.6936	0.0005	(0.0504,3.0006)

Table (2): *Biases, MSE and RE of the $\hat{\psi}^{MLE}_{n_1,n_2}$ and $\hat{\psi}^{Bayes}_{n_1,n_2}$*

$(\boldsymbol{\alpha},\boldsymbol{\beta_1},\boldsymbol{\beta_2}) = (\mathbf{16},\mathbf{2.0056},\mathbf{3.0614})$														
$(\boldsymbol{n_1},\boldsymbol{n_2})$	$(\boldsymbol{n},\boldsymbol{m})$	ψ_{n_1,n_2}	$\hat{\psi}^{MLE}_{n_1,n_2}$	$\hat{\psi}^{Bayes}_{n_1,n_2}$	$\hat{\psi}^{Bayes\ MCMC}_{n_1,n_2}$	**MLE**		**Bayes** $\hat{\psi}^{Bayes}_{n_1,n_2}$		**Bayes** $\hat{\psi}^{Bayes\ MCMC}_{n_1,n_2}$		**RE1**	**RE2**	**RE3**
						Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**			
(5,5)	(10,10)	0.9554	0.9598	1.0843	1.2921	0.0004	$1.93*10^{-6}$	0.0124	0.0017	0.0336	0.0113	0.0012	0.0016	0.0150
	(20,20)		0.9546	1.0374	1.2883	-0.0004	$3*10^{-8}$	0.0041	0.0033	0.0166	0.0055	0.0001	$5*10^{-6}$	0.6000
	(30,30)		0.9586	1.0393	1.2860	0.0001	$3.4*10^{-7}$	0.0003	0.0002	0.0110	0.0036	0.0014	$2.4*10^{-4}$	0.0555
	(40,40)		0.9575	1.0469	1.2902	0.0001	$1.1*10^{-7}$	0.0022	0.0003	0.0083	0.0028	0.0005	$0.22*10^{-3}$	0.6000
	(50,50)		0.9504	1.0137	1.2802	-0.0001	$5*10^{-7}$	0.0013	0.0001	0.0064	0.0021	0.0073	$6.8*10^{-5}$	0.0500
(10,10)	(10,10)	0.9631	0.9636	1.1062	1.3382	0.0001	$2.5*10^{-8}$	0.0143	0.0020	0.0375	0.0141	0.00001	$0.1*10^{-5}$	0.1418
	(20,20)		0.9601	1.0546	1.3340	-0.0002	$4.5*10^{-7}$	0.0047	0.0004	0.0185	0.0068	0.0011	$0.67*10^{-4}$	0.0588
	(30,30)		0.9640	1.0383	1.3315	-0.00003	$2.7*10^{-8}$	0.0024	0.0002	0.0122	0.0045	0.0001	$0.6*10^{-5}$	0.0444
	(40,40)		0.9629	1.0272	1.3367	0.0001	$0.1*10^{-8}$	0.0016	0.0001	0.0093	0.0034	0.00001	$0.3*10^{-6}$	0.0294
	(50,50)		0.946	1.0264	1.3244	0.00003	$4.5*10^{-8}$	0.0012	$0.8*10^{-4}$	0.0072	0.0026	0.0006	$0.17*10^{-4}$	0.0307

Table (3): *ACL and for CP 99% ACI and BCI for ψ_{n_1,n_2}*

$(\boldsymbol{\alpha},\boldsymbol{\beta_1},\boldsymbol{\beta_2}) = (\mathbf{16},\mathbf{2.0056},\mathbf{3.0614})$					
$(\boldsymbol{n_1},\boldsymbol{n_2})$	$(\boldsymbol{n},\boldsymbol{m})$	**MLE**		**BCI**	
		ACL	**CP**	**ACL**	**CP**
(5,5)	(10,10)	0.0097	0.997	0.0132	0.995
	(20,20)	0.0173	0.995	0.0014	0.996
	(30,30)	0.0152	0.996	0.0119	0.995

	(40,40)	0.0157	0.996	0.0099	0.996
	(50,50)	0.0168	0.995	0.0103	0.996
(10,10)	(10,10)	0.0208	0.994	0.0162	0.993
	(20,20)	0.0201	0.995	0.0078	0.997
	(30,30)	0.0149	0.996	0.0082	0.997
	(40,40)	0.0149	0.996	0.0077	0.997
	(50,50)	0.137	0.996	0.0088	0.997

Table 4: Results of Goodness-of-fit Tests

Data Set	MLE	K-S(p-value)	AD(p-value)
Data set I	(1.1248,672.35)	0.911(0.968)	0.231(0.981)
Data set II	(1.1398,560.08)	0.096(0.949)	0.254(0.969)

* (k-S): Kolmogorov–Smirnov
* AD: Anderson–Darling

Table 5: *Results of data sets of number of tumours treated in breast cancer disease.*

(n_1, n_2)	$\hat{\boldsymbol{\psi}}^{\mathbf{MLE}}_{\mathbf{n_1,n_2}}$	$\hat{\boldsymbol{\psi}}^{\mathbf{Bayes}}_{\mathbf{n_1,n_2}}$	$\hat{\psi}^{Bayes\ MCMC}_{n_1,n_2}$	ACI
(5,5)	0.8969	0.9954	1.328	(0.8964,0.8973)
(10,10)	0.0524	0.1045	0.1331	(0.0512,0.0645)

Abbreviations

MRS	Mean Remaining Strength	RE	Relative Efficiency
AC I	Asymptotic Confidence Interval	BC I	Bootstrap Confidence Interval
MLE	Maximum Likelihood Estimator	ACL	Average Confidence Length
MSE	Mean Square Error	CP	Converge Probabilities
CI	Confidence Interval		