



New Bivariate Transmuted Rayleigh Distribution

Marwa KH. Hassan ^{1*}

¹Department of Mathematics, Faculty of Education, Ain Shams University, Cairo 11566, Egypt

* Corresponding Email: marwa_khalil@edu.asu.edu.eg

Received: 22 July 2026 / Revised: 20 September 2026 / Accepted: 29 September 2026 / Published online: 11 October 2026

This is an Open Access article published under the Creative Commons Attribution 4.0 International (CC BY 4.0) (<https://creativecommons.org/licenses/by/4.0/>). © SCOPUA Journal of Applied Statistical Research published by SCOPUA (Scientific Collaborative Online Publishing Universal Academy). SCOPUA stands neutral with regard to jurisdictional claims in the published maps and institutional affiliations.

ABSTRACT

The Rayleigh distribution serves as a model for lifetime events and finds numerous applications in survival analysis, reliability, communications, and various other fields. Many bivariate situations require explanation, making bivariate distributions crucial for describing these scenarios. In this paper, we address this issue by introducing a new bivariate distribution known as the bivariate transmuted Rayleigh distribution. We present an analytical study of this distribution, which encompasses its properties, inference, and real-life applications.

Keywords: Transmuted Distribution; Bivariate Distribution; Estimation Method; Rayleigh Distribution; Simulation
MSC: 62E15; 62F10; 62H10; 62H05

1. Introduction

The issue of bivariate distributions arises when examining the relationship between two variables. In statistics, bivariate distributions are crucial for analyzing the joint behavior of two variables, particularly in correlation and regression analysis. The primary significance of bivariate distributions in statistics lies in elucidating the relationship between two variables, their mutual effects, and the explanation of complex models. Bivariate distributions are utilized across various fields, including data analysis, machine learning, and econometrics. Consequently, numerous authors have contributed to the development of transmuted distributions. [1] introduced bivariate distributions with transmuted univariate marginals. The bivariate generalized transmuted G distribution was presented by [2]. [3] introduced the generalized transmuted distribution. [4] presented the exponentiated transmuted G distribution, while [5] introduced a bivariate generalized exponentiated distribution. [6] presented a bivariate transmuted general family distribution. [7] introduced a bivariate extension of the univariate transmuted distribution family, and [8] presented a bivariate transmuted Burr distribution.

Definition 1: The joint cumulative distribution function of bivariate extension of the T-X family of distributions is defined as:

$$F_{x,y}(x, y) = \int_{a_2}^{W_2[G_2(y)]} \int_{a_1}^{W_1[G_1(x)]} r(u_1, u_2) du_1 du_2$$

Where, $W_1[G_1(x)]$ and $W_2[G_2(y)]$ have the same properties and $r(u_1, u_2)$ is a bivariate distribution. [8] introduced a simple version of the bivariate T-X family of distributions when u_1 and u_2 have support $[0,1]$ as:

$$F_{x,y}(x, y) = \int_0^{G_2(y)} \int_0^{G_1(x)} r(u_1, u_2) du_1 du_2$$

And they used $r(u_1, u_2) = 1 + \lambda_1 (1 - 2 u_1) + \lambda_2 (1 - 2 u_2) + 2 \lambda_3 (1 - u_1 - u_2)$,

Then, $F_{x,y}(x, y)$ in this case is given by,

$$F_{x,y}(x, y) = G_1(x) G_2(y) (1 + (\lambda_1 + \lambda_3)(1 - G_1(x)) + (\lambda_2 + \lambda_3)(1 - G_2(y))), (x, y) \in R^2.$$



Where, $G_1(x)$ and $G_2(y)$ are cdfs of any baseline distribution and λ_1, λ_2 and λ_3 are transmutation parameters such that $(\lambda_1, \lambda_2, \lambda_3) \in [-1, 1]$, $-1 \leq \lambda_1 + \lambda_3 \leq 1$ and $-1 \leq \lambda_2 + \lambda_3 \leq 1$. In this case the corresponding pdf is given by,

$$f_{x,y}(x, y) = g_1(x) g_2(y) (1 + (\lambda_1 + \lambda_3)(1 - 2G_1(x)) + (\lambda_2 + \lambda_3)(1 - 2G_2(y))), (x, y) \in R^2$$

Where, $g_1(x)$ and $g_2(y)$ are pdfs for $G_1(x)$ and $G_2(y)$.

The transmuted family has been utilized by numerous authors. For the transmuted Rayleigh distribution, see [9]. The motivation of this paper is to extend the transmuted Rayleigh distribution to the bivariate case, which we refer to as the bivariate transmuted Rayleigh distribution (BTRD). The choice of the Rayleigh distribution is due to its status as a lifetime distribution with many applications in survival analysis, reliability, communications, and various other fields. This construction was selected because it provides a flexible and analytically tractable framework for developing a bivariate distribution from specified univariate marginal distributions while allowing dependence between the two components to be incorporated in a relatively simple form. It enables the proposed BTRD to retain its marginal distributional characteristics while introducing a dependence parameter that controls the association between the two random variables.

Several alternative bivariate construction methods are available in the literature, including the Farlie–Gumbel–Morgenstern (FGM) family, Archimedean copulas, and the general bivariate T-X family. Although these approaches are useful in many applications, they may impose different restrictions on the attainable dependence structure, or lead to more complicated analytical expressions for the resulting joint density and distribution functions. For example, the FGM family is attractive because of its simplicity and closed-form representation, but it is known to provide only a limited range of dependence. Archimedean copulas offer greater flexibility in dependence modeling, but their associated joint distributions and likelihood functions can become considerably more complicated depending on the selected generator. Similarly, the general bivariate T-X construction is highly flexible but may introduce additional mathematical complexity and parameters.

This paper is organized as follows. Section 2 introduces the BTRD, while Section 3 presents its main statistical properties. The estimation of the distribution parameters using the maximum likelihood method is discussed in Section 4. Section 5 presents a comprehensive simulation study designed to investigate the performance and behavior of the maximum likelihood estimators. Section 6 demonstrates the practical applicability of the BTRD through a real-data application. Finally, Section 7 summarizes the main findings and presents the conclusions of the study.

2. Bivariate Transmuted Rayleigh Distribution (BTRD)

[10] introduced the Rayleigh distribution as a special case of Weibull distribution with shape parameter equal 2.

Definition 2: The probability density function (pdf) of Rayleigh distribution is.

$$f(x; \sigma) = \frac{x}{\sigma^2} e^{-\left(\frac{x^2}{2\sigma^2}\right)}, \quad x > 0, \sigma > 0. \quad (1)$$

And, the cumulative distribution function (cdf) of Rayleigh distribution is.

$$F(x; \sigma) = 1 - e^{-\left(\frac{x^2}{2\sigma^2}\right)}, \quad x > 0, \sigma > 0. \quad (2)$$

Figure (1) shows the pdf and cdf of Rayleigh distribution of different values of σ .

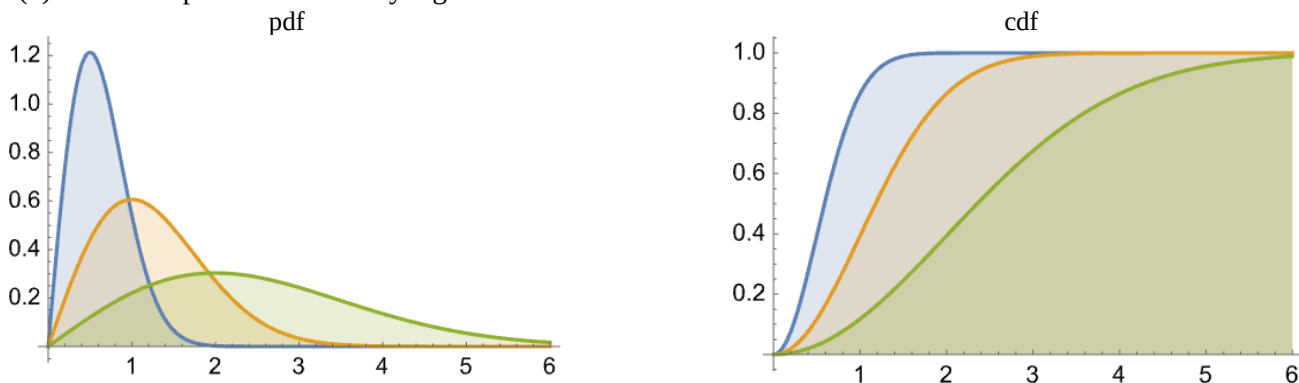


Figure (1): The pdf and cdf of Rayleigh distribution of different values of $\sigma = (0.5, 1, 2)$.

[9] introduced the transmuted Rayleigh distribution as follows:

Definition 3: The probability density function (pdf) of the transmuted Rayleigh distribution

$$f(x; \sigma, \lambda) = \frac{x}{\sigma^2} e^{-\left(\frac{x^2}{2\sigma^2}\right)} (1 - \lambda + 2\lambda e^{-\left(\frac{x^2}{2\sigma^2}\right)}), \quad x > 0, \sigma > 0, -1 \leq \lambda \leq 1. \quad (3)$$

And, the cumulative distribution function (cdf) of the transmuted Rayleigh distribution is.

$$F(x; \sigma, \lambda) = \left(1 - e^{-\left(\frac{x^2}{2\sigma^2}\right)}\right) (1 + \lambda e^{-\left(\frac{x^2}{2\sigma^2}\right)}), \quad x > 0, \sigma > 0, -1 \leq \lambda \leq 1. \quad (4)$$

Which is an extended model to analyze more complex data. If $\lambda = 0$, we get the Rayleigh distribution. Figure (2) shows the pdf and cdf of the transmuted Rayleigh distribution of different values of σ and λ .

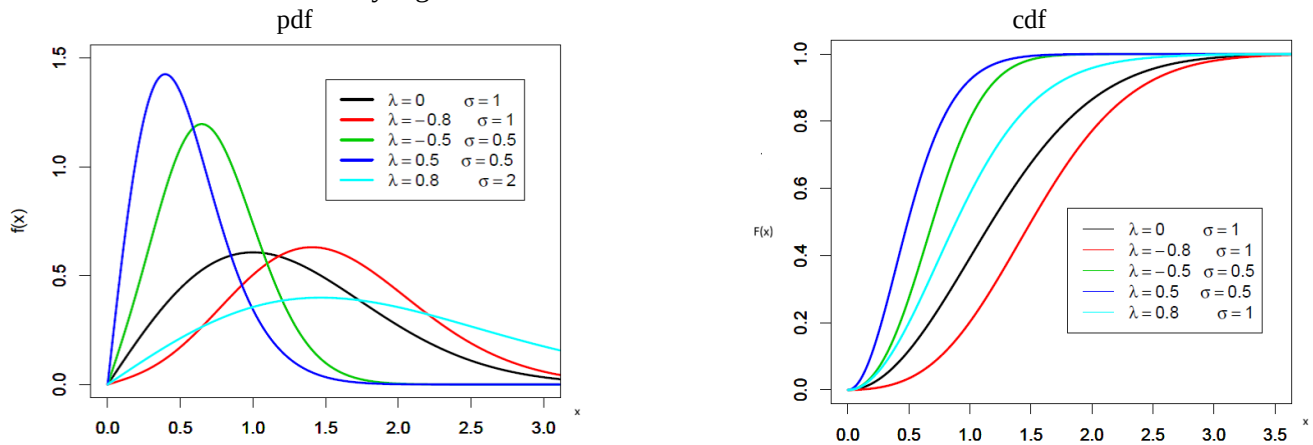


Figure (2): The pdf and cdf of the transmuted Rayleigh distribution of different values of σ, λ .

In this paper, we present the bivariate transmuted Rayleigh distribution (BTRD), which is utilized to describe more complex phenomena involving two variables.

Definition 4: The joint cumulative distribution function (jcdf) of the bivariate transmuted Rayleigh distribution (BTRD) is,

$$F_{X,Y}(x, y) = \left(1 - e^{-\frac{x^2}{2\sigma_1^2}}\right) \left(1 - e^{-\frac{y^2}{2\sigma_2^2}}\right) \left(1 + \alpha e^{-\frac{x^2}{2\sigma_1^2}} + \beta e^{-\frac{y^2}{2\sigma_2^2}}\right),$$

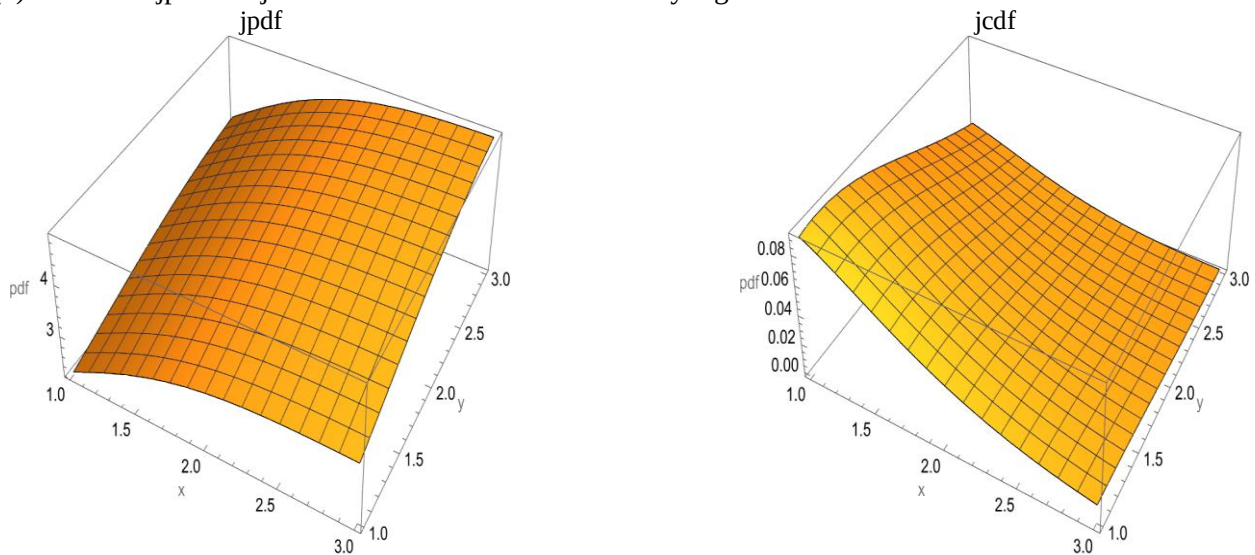
$$\alpha = \lambda_1 + \lambda_3 \text{ and } \beta = \lambda_2 + \lambda_3, x, y > 0, -1 \leq \lambda_1, \lambda_2, \lambda_3 \leq 1. \quad (5)$$

And the joint probability density function (jpdf) of the bivariate transmuted Rayleigh distribution (BTRD) is,

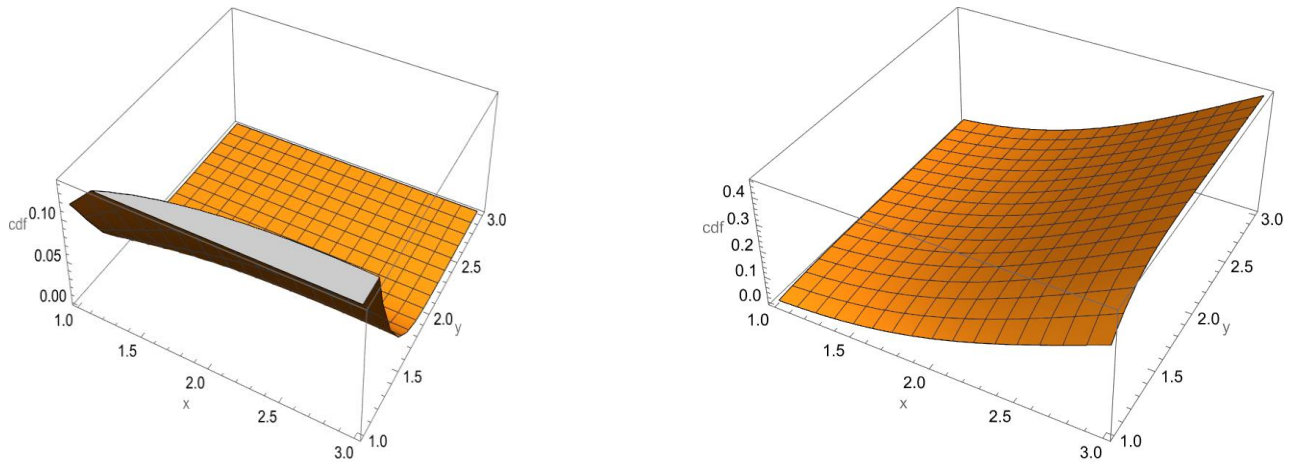
$$f_{X,Y}(x, y) = \frac{xy}{\sigma_1^2 \sigma_2^2} e^{-\frac{x^2}{2\sigma_1^2} - \frac{y^2}{2\sigma_2^2}} \left(1 + \alpha \left(2e^{-\frac{x^2}{2\sigma_1^2}} - 1\right) + \beta \left(2e^{-\frac{y^2}{2\sigma_2^2}} - 1\right)\right)$$

$$\alpha = \lambda_1 + \lambda_3 \text{ and } \beta = \lambda_2 + \lambda_3, x, y > 0, -1 \leq \lambda_1, \lambda_2, \lambda_3 \leq 1. \quad (6)$$

Figure (3) shows the jpdf and jcdf of the bivariate transmuted Rayleigh distribution.



$(\sigma_1, \sigma_2, \lambda_1, \lambda_2, \lambda_3) = (1.4, 2.0, 0.5, 0.5, 0.5)$



$(\sigma_1, \sigma_2, \lambda_1, \lambda_2, \lambda_3) = (2, 0.5, -0.5, -0.5, -0.5)$

Figure (3): The jpdf and jcdf of the bivariate transmuted Rayleigh distribution

3. Statistical Properties of BTRD

We discussed several statistical properties of the newly proposed distribution, including marginal and conditional distributions, product and ratio moments, conditional moments, bivariate reliability, bivariate hazard functions, the stress-strength model, dependence function, Clayton-Oakes association measure, and conditional probability measure.

a. The Marginal and Conditional distribution of BTRD

The marginal probability density function of X and Y respectively are given by,

$$f_X(x) = \int_0^\infty f_{X,Y}(x, y) dy.$$

$$= \frac{e^{-\frac{x^2}{\sigma_1^2}} (-e^{-\frac{x^2}{2\sigma_1^2}(\lambda_1 + \lambda_3 - 1) + 2(\lambda_1 + \lambda_3)x})}{\sigma_1^2}, \quad x > 0, \sigma_1 > 0, 1 \leq \lambda_1, \lambda_2, \lambda_3 \leq 1 \quad (7)$$

And,

$$f_Y(y) = \int_0^\infty f_{X,Y}(x, y) dx$$

$$= \frac{e^{-\frac{y^2}{\sigma_2^2}} (-e^{-\frac{y^2}{2\sigma_2^2}(\lambda_2 + \lambda_3 - 1) + 2(\lambda_2 + \lambda_3)y})}{\sigma_2^2}, \quad x > 0, \sigma_1 > 0, -1 \leq \lambda_1, \lambda_2, \lambda_3 \leq 1 \quad (8)$$

The marginal cumulative distribution functions of X and Y respectively are given by,

$$F_X(x) = \int_0^\infty F_{X,Y}(x, y) dy$$

$$= \frac{(1 - e^{-\frac{x^2}{2\sigma_1^2}}) \sqrt{\pi} (-\sqrt{2} + (-1 + \sqrt{2})\lambda_2 + (-1 + \sqrt{2})\lambda_3 - \sqrt{2}e^{-\frac{x^2}{2\sigma_1^2}(\lambda_1 + \lambda_3)})}{2\sqrt{\frac{1}{\sigma_1^2}}}. \quad (9)$$

And,

$$F_Y(y) = \int_0^\infty F_{X,Y}(x, y) dx$$

$$= \frac{(1 - e^{-\frac{y^2}{2\sigma_2^2}}) \sqrt{\pi} (-\sqrt{2} + (-1 + \sqrt{2})\lambda_1 + (-1 + \sqrt{2})\lambda_3 - \sqrt{2}e^{-\frac{y^2}{2\sigma_2^2}(\lambda_2 + \lambda_3)})}{2\sqrt{\frac{1}{\sigma_2^2}}}. \quad (10)$$

The conditional probability density function of X given Y and Y given X are given respectively as

$$f(X|Y) = \frac{e^{-\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2}} \left(1 + \left(-1 + 2e^{-\frac{x^2}{2\sigma_1^2}} \right) (\lambda_1 + \lambda_3) + \left(-1 + 2e^{-\frac{y^2}{2\sigma_2^2}} \right) (\lambda_2 + \lambda_3) \right) y}{\sigma_2^2 \left(-e^{-\frac{x^2}{2\sigma_1^2}(-1 + \lambda_1 + \lambda_3) + 2(\lambda_1 + \lambda_3)x} \right)}. \quad (11)$$

And,

$$f(Y|X) = \frac{e^{-\frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2}} \left(1 + \left(-1 + 2e^{-\frac{x^2}{2\sigma_1^2}} \right) (\lambda_1 + \lambda_3) + \left(-1 + 2e^{-\frac{y^2}{2\sigma_2^2}} \right) (\lambda_2 + \lambda_3) \right) x}{\sigma_1^2 \left(-e^{-\frac{y^2}{2\sigma_2^2}} (-1 + \lambda_2 + \lambda_3) + 2(\lambda_2 + \lambda_3) \right)} \quad (12)$$

b. Moments of BTRD

We derived two types of moments for BTRD, the first type is the conditional moment of X given Y=y which is defined as,

$$E(X^r|y) = \frac{\left(\frac{1}{\sigma_1^2} \right)^{-r/2} r \left(2^{1+\frac{r}{2}} (\lambda_2 + \lambda_3) + e^{\frac{y^2}{2\sigma_2^2}} (\lambda_1 + \lambda_3 - 2^{r/2} (-1 + \lambda_1 + \lambda_2 + 2\lambda_3)) \right) \Gamma\left[\frac{r}{2}\right]}{2 \left(-e^{\frac{y^2}{2\sigma_2^2}} (-1 + \lambda_2 + \lambda_3) + 2(\lambda_2 + \lambda_3) \right)} \quad (13)$$

The conditional moment of Y given X=x which is defined as,

$$E[Y^s|x] = \frac{\left(\frac{1}{\sigma_2^2} \right)^{-s/2} s \left(2^{1+\frac{s}{2}} (\lambda_1 + \lambda_3) + e^{\frac{x^2}{2\sigma_1^2}} (\lambda_2 + \lambda_3 - 2^{s/2} (-1 + \lambda_1 + \lambda_2 + 2\lambda_3)) \right) \Gamma\left[\frac{s}{2}\right]}{2 \left(-e^{\frac{x^2}{2\sigma_1^2}} (-1 + \lambda_1 + \lambda_3) + 2(\lambda_1 + \lambda_3) \right)} \quad (14)$$

The second type is product moments which are useful in measuring the dependence between two variables.

$$E[X^r Y^s] = \int_0^\infty \int_0^\infty x^r y^s f_{X,Y}(x, y) dx dy$$

$$= \frac{1}{4} \left(\frac{1}{\sigma_1^2} \right)^{-r/2} \left(\frac{1}{\sigma_2^2} \right)^{-s/2} s r (2^{h/2} (\lambda_1 + \lambda_3) + 2^{r/2} (\lambda_2 + \lambda_3 - 2^{s/2} (-1 + \lambda_1 + \lambda_2 + 2\lambda_3))) \Gamma\left[\frac{s}{2}\right] \Gamma\left[\frac{r}{2}\right] \quad (15)$$

c. The Reliability and Hazard Rate function

The reliability function is called also the survival function. This is crucial in many fields such as reliability engineering, survival analysis and so on. It is defined as the bivariate distribution as follows, $R_{X,Y}(x, y) = 1 - F_X(x) - F_Y(y) - F_{X,Y}(x, y)$ and it is defined by the new distribution BTRD as,

$$R_{X,Y}(x, y) = 1 - \frac{\left(1 - e^{-\frac{x^2}{2\sigma_1^2}} \right) \sqrt{\pi} \left(-\sqrt{2} + (-1 + \sqrt{2})\lambda_2 + (-1 + \sqrt{2})\lambda_3 - \sqrt{2} e^{-\frac{x^2}{2\sigma_1^2}} (\lambda_1 + \lambda_3) \right)}{2 \sqrt{\frac{1}{\sigma_2^2}}} -$$

$$\frac{\left(1 - e^{-\frac{y^2}{2\sigma_2^2}} \right) \sqrt{\pi} \left(-\sqrt{2} + (-1 + \sqrt{2})\lambda_1 + (-1 + \sqrt{2})\lambda_3 - \sqrt{2} e^{-\frac{y^2}{2\sigma_2^2}} (\lambda_2 + \lambda_3) \right)}{2 \sqrt{\frac{1}{\sigma_1^2}}} +$$

$$\left(1 - e^{-\frac{x^2}{2\sigma_1^2}} \right) \left(1 - e^{-\frac{y^2}{2\sigma_2^2}} \right) \left(1 + (\lambda_1 + \lambda_3) e^{-\frac{x^2}{2\sigma_1^2}} + (\lambda_2 + \lambda_3) e^{-\frac{y^2}{2\sigma_2^2}} \right) \quad (16)$$

Also, we introduced the bivariate hazard rate function which is important to survival analysis because it is refer to the rate of death for any items. The hazard function for BTRD is defined by,

$$h_{X,Y}(x, y) = \frac{f_{X,Y}(x, y)}{R_{X,Y}(x, y)}$$

$$= \left(\frac{x}{\sigma_1^2} e^{-\frac{x^2}{2\sigma_1^2}} \right) \left(\frac{y}{\sigma_2^2} e^{-\frac{y^2}{2\sigma_2^2}} \right) \left(1 + (\lambda_1 + \lambda_3) \left(2 e^{-\frac{x^2}{2\sigma_1^2}} - 1 \right) + (\lambda_2 + \lambda_3) \left(2 e^{-\frac{y^2}{2\sigma_2^2}} - 1 \right) \right) /$$

$$\left(1 - e^{-\frac{x^2}{2\sigma_1^2}} \right) \sqrt{\pi} \left(-\sqrt{2} + (-1 + \sqrt{2})\lambda_2 + (-1 + \sqrt{2})\lambda_3 - \sqrt{2} e^{-\frac{x^2}{2\sigma_1^2}} (\lambda_1 + \lambda_3) \right)$$

$$\left(1 - e^{-\frac{y^2}{2\sigma_2^2}} \right) \sqrt{\pi} \left(-\sqrt{2} + (-1 + \sqrt{2})\lambda_1 + (-1 + \sqrt{2})\lambda_3 - \sqrt{2} e^{-\frac{y^2}{2\sigma_2^2}} (\lambda_2 + \lambda_3) \right) +$$

$$\left(1 - e^{-\frac{x^2}{2\sigma_1^2}} \right) \left(1 - e^{-\frac{y^2}{2\sigma_2^2}} \right) \left(1 + (\lambda_1 + \lambda_3) e^{-\frac{x^2}{2\sigma_1^2}} + (\lambda_2 + \lambda_3) e^{-\frac{y^2}{2\sigma_2^2}} \right) \quad (17)$$



$$\left(1 - e^{-\frac{x^2}{2\sigma_1^2}}\right)\left(1 - e^{-\frac{y^2}{2\sigma_2^2}}\right)\left(1 + (\lambda_1 + \lambda_3)e^{-\frac{x^2}{2\sigma_1^2}} + (\lambda_2 + \lambda_3)e^{-\frac{y^2}{2\sigma_2^2}}\right) \quad (17)$$

For more information on reliability and hazard rate function see [11] and [12].

d. Correlation Coefficients

[8] introduced the two correlation coefficients to measure the dependence between two variables.

Definition 5: The Kendall's tau coefficient for two variables is defined as follows,

$$\tau = 4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_{X,Y}(x, y) f_{X,Y}(x, y) dx dy - 1$$

In the case of the bivariate transmuted distributions is defined as,

$$\tau = -\frac{2}{9} (\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3).$$

Definition 6: The Spearman's rho for two variables is defined as follows,

$$\rho = 12 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (F_{X,Y}(x, y) - F_X(x) F_Y(y)) f_X(x) f_Y(y) dx dy$$

In the case of the bivariate transmuted distributions is defined as,

$$\rho = -\frac{1}{3} (\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3).$$

e. The Stress-Strength Model

The stress-strength model, which describes the lifetime of a component with strength random variable Y subject to the stress random variable X. the stress-strength model of BTRD is defined as ,

$$R = P[X < Y] = \int_0^{\infty} \int_0^y f(x, y) dx dy, \\ = \frac{2\sigma_2^6 + \sigma_1^2 \sigma_2^4 (5 + \lambda_1 - 2\lambda_2 - \lambda_3) + \sigma_1^4 \sigma_2^2 (2 + 2\lambda_1 - \lambda_2 + \lambda_3)}{(\sigma_1^2 + \sigma_2^2)(2\sigma_1^2 + \sigma_2^2)(\sigma_1^2 + 2\sigma_2^2)}. \quad (18)$$

For more information about stress-strength model of bivariate distribution see [13].

2.6 Dependence

[14] and [15] are defined the local dependence function $\gamma(x, y)$ which is defined as,

$$\gamma(x, y) = \frac{\partial^2}{\partial x \partial y} \ln f_{X,Y}(x, y).$$

In the case of BTRD is defined as,

$$\gamma(x, y) = -\frac{4e^{\frac{1}{2}(\frac{x^2}{\sigma_1^2} + \frac{y^2}{\sigma_2^2})} (\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)xy}{\sigma_1^2 \sigma_2^2 (2e^{-\frac{x^2}{2\sigma_1^2}}(\lambda_2 + \lambda_3) + e^{-\frac{y^2}{2\sigma_2^2}}(2(\lambda_1 + \lambda_3) - e^{-\frac{x^2}{2\sigma_1^2}}(-1 + \lambda_1 + \lambda_2 + 2\lambda_3)))^2}. \quad (19)$$

f. Clayton–Oakes Association Measure

[16] and [17] are defined as a local dependence function as follows:

$$l(x, y) = \frac{f_{X,Y}(x, y) R_{X,Y}(x, y)}{R_X(x, y) R_Y(x, y)},$$

$$\text{where } R_X(x, y) = \frac{\partial}{\partial x} R_{X,Y}(x, y) \text{ and } R_Y(x, y) = \frac{\partial}{\partial y} R_{X,Y}(x, y).$$

In the case of BTRD is defined as,

$$l(x, y) = \frac{(4(1 + (-1 + 2e^{-\frac{x^2}{2\sigma_1^2}})(\lambda_1 + \lambda_3) + (-1 + 2e^{-\frac{y^2}{2\sigma_2^2}})(\lambda_2 + \lambda_3))(1 - e^{-\frac{x^2}{2\sigma_1^2}})\sqrt{\pi}(-\sqrt{2} + (-1 + \sqrt{2})\lambda_2 + (-1 + \sqrt{2})\lambda_3 - \sqrt{2}e^{-\frac{x^2}{2\sigma_1^2}}(\lambda_1 + \lambda_3))}{2\sqrt{\frac{1}{\sigma_2^2}}} + \frac{(1 - e^{-\frac{y^2}{2\sigma_2^2}})\sqrt{\pi}(-\sqrt{2} + (-1 + \sqrt{2})\lambda_1 + (-1 + \sqrt{2})\lambda_3 - \sqrt{2}e^{-\frac{y^2}{2\sigma_2^2}}(\lambda_2 + \lambda_3))}{2\sqrt{\frac{1}{\sigma_1^2}}})}{\left(1 - e^{-\frac{x^2}{2\sigma_1^2}}\right)\left(1 - e^{-\frac{y^2}{2\sigma_2^2}}\right)\left(1 + e^{-\frac{x^2}{2\sigma_1^2}}(\lambda_1 + \lambda_3) + e^{-\frac{y^2}{2\sigma_2^2}}(\lambda_2 + \lambda_3)\right) - \frac{(1 - e^{-\frac{y^2}{2\sigma_2^2}})\sqrt{\pi}(-\sqrt{2} + (-1 + \sqrt{2})\lambda_1 + (-1 + \sqrt{2})\lambda_3 - \sqrt{2}e^{-\frac{y^2}{2\sigma_2^2}}(\lambda_2 + \lambda_3))}{2\sqrt{\frac{1}{\sigma_1^2}}})}.$$

$$\begin{aligned}
& ((-2(1 - e^{-\frac{x^2}{2\sigma_1^2}})(1 - e^{-\frac{y^2}{2\sigma_2^2}})(\lambda_1 + \lambda_3) - \frac{(1 - e^{-\frac{x^2}{2\sigma_1^2}})\sqrt{2\pi}(\lambda_1 + \lambda_3)}{\sqrt{\frac{1}{\sigma_2^2}}}) \\
& + \frac{\sqrt{\pi}(\sqrt{2} + \lambda_2 - \sqrt{2}\lambda_2 + \lambda_3 - \sqrt{2}\lambda_3 + \sqrt{2}e^{-\frac{x^2}{2\sigma_1^2}}(\lambda_1 + \lambda_3))}{\sqrt{\frac{1}{\sigma_2^2}}}) \\
& + 2(1 - e^{-\frac{y^2}{2\sigma_2^2}})(1 + e^{-\frac{x^2}{2\sigma_1^2}}(\lambda_1 + \lambda_3) + e^{-\frac{y^2}{2\sigma_2^2}}(\lambda_2 + \lambda_3)))(-2(1 - e^{-\frac{x^2}{2\sigma_1^2}})(1 - e^{-\frac{y^2}{2\sigma_2^2}})(\lambda_2 + \lambda_3) - \\
& \frac{(1 - e^{-\frac{y^2}{2\sigma_2^2}})\sqrt{2\pi}(\lambda_2 + \lambda_3)}{\sqrt{\frac{1}{\sigma_1^2}}} + 2(1 - e^{-\frac{x^2}{2\sigma_1^2}})(1 + e^{-\frac{x^2}{2\sigma_1^2}}(\lambda_1 + \lambda_3) + e^{-\frac{y^2}{2\sigma_2^2}}(\lambda_2 + \lambda_3)) + \\
& \frac{\sqrt{\pi}(\sqrt{2} + \lambda_1 - \sqrt{2}\lambda_1 + \lambda_3 - \sqrt{2}\lambda_3 + \sqrt{2}e^{-\frac{y^2}{2\sigma_2^2}}(\lambda_2 + \lambda_3))}{\sqrt{\frac{1}{\sigma_1^2}})))). \tag{20}
\end{aligned}$$

For more information about this measure see [18].

g. Conditional Probability Measure

[19] are defined as a measure of association based on conditional probability as,

$$\begin{aligned}
\Psi(x, y) &= \frac{P[X > x | Y > y]}{P[X | x]}, \\
&= \frac{R_{X,Y}(x, y)}{R_{X,Y}(0, y) R_{X,Y}(x, 0)},
\end{aligned}$$

In the case of BTRD it is defined as,

$$\begin{aligned}
\Psi(x, y) &= (1 - \frac{(1 - e^{-\frac{x^2}{2\sigma_1^2}})\sqrt{\pi}(-\sqrt{2} + (-1 + \sqrt{2})\lambda_2 + (-1 + \sqrt{2})\lambda_3 - \sqrt{2}e^{-\frac{x^2}{2\sigma_1^2}}(\lambda_1 + \lambda_3))}{2\sqrt{\frac{1}{\sigma_2^2}}}) \\
& + (1 - e^{-\frac{x^2}{2\sigma_1^2}})(1 - e^{-\frac{y^2}{2\sigma_2^2}})(1 + e^{-\frac{x^2}{2\sigma_1^2}}(\lambda_1 + \lambda_3) + e^{-\frac{y^2}{2\sigma_2^2}}(\lambda_2 + \lambda_3)) - \\
& \frac{(1 - e^{-\frac{y^2}{2\sigma_2^2}})\sqrt{\pi}(-\sqrt{2} + (-1 + \sqrt{2})\lambda_1 + (-1 + \sqrt{2})\lambda_3 - \sqrt{2}e^{-\frac{y^2}{2\sigma_2^2}}(\lambda_2 + \lambda_3))}{2\sqrt{\frac{1}{\sigma_1^2}}}) / \\
& ((1 - \frac{(1 - e^{-\frac{x^2}{2\sigma_1^2}})\sqrt{\pi}(-\sqrt{2} + (-1 + \sqrt{2})\lambda_2 + (-1 + \sqrt{2})\lambda_3 - \sqrt{2}e^{-\frac{x^2}{2\sigma_1^2}}(\lambda_1 + \lambda_3))}{2\sqrt{\frac{1}{\sigma_2^2}}}) \\
& (1 - \frac{(1 - e^{-\frac{y^2}{2\sigma_2^2}})\sqrt{\pi}(-\sqrt{2} + (-1 + \sqrt{2})\lambda_1 + (-1 + \sqrt{2})\lambda_3 - \sqrt{2}e^{-\frac{y^2}{2\sigma_2^2}}(\lambda_2 + \lambda_3))}{2\sqrt{\frac{1}{\sigma_1^2}}})))). \tag{21}
\end{aligned}$$

4. Estimation Method for parameters of BTRD

In this section, we used maximum likelihood estimation method to get the maximum likelihood estimators of BTRD as follows: Let $(x_1, y_1) \dots (x_n, y_n)$ be a random sample of size n from BTRD, then the likelihood function can be defined as:

$$L(\underline{\theta}) = \prod_{i=1}^n f_{X,Y}(x_i, y_i),$$

$$= \prod_{i=1}^n \left(\frac{x_i}{\sigma_1^2} e^{-\frac{x_i^2}{2\sigma_1^2}} \right) \left(\frac{y_i}{\sigma_2^2} e^{-\frac{y_i^2}{2\sigma_2^2}} \right) \left(1 + (\lambda_1 + \lambda_3) \left(2e^{-\frac{x_i^2}{2\sigma_1^2}} - 1 \right) + (\lambda_2 + \lambda_3) \left(2e^{-\frac{y_i^2}{2\sigma_2^2}} - 1 \right) \right)$$

And, the log-likelihood function is given as,

$$l(\underline{\theta}) = (Log[L(\underline{\theta})]) = -n(Log[\sigma_1^2] + Log[\sigma_2^2]) - \frac{1}{2} \sum_{i=1}^n \left(\frac{x_i^2}{\sigma_1^2} + \frac{y_i^2}{\sigma_2^2} \right) +$$

$$\sum_{i=1}^n \text{Log} \left[(1 + (\lambda_1 + \lambda_3) \left(2 e^{-\frac{x_i^2}{2\sigma_1^2}} - 1 \right) + (\lambda_2 + \lambda_3) \left(2 e^{-\frac{y_i^2}{2\sigma_2^2}} - 1 \right) \right].$$

Where, $\underline{\Theta} = (\sigma_1, \sigma_2, \lambda_1, \lambda_2, \lambda_3)$.

To get the maximum likelihood estimators of the distribution parameters, we get the derivatives of unknown parameters,

$$\begin{aligned} \frac{\partial}{\partial \sigma_1} \text{Log}[L(\underline{\Theta})] &= -\frac{2n}{\sigma_1} + \sum_{i=1}^n \frac{x_i^2}{\sigma_1^3} + \sum_{i=1}^n \frac{2e^{-\frac{x_i^2}{2\sigma_1^2}}(\lambda_1 + \lambda_3)x_i^2}{\sigma_1^3(1 + (-1 + 2e^{-\frac{x_i^2}{2\sigma_1^2}})(\lambda_1 + \lambda_3) + (-1 + 2e^{-\frac{y_i^2}{2\sigma_2^2}})(\lambda_2 + \lambda_3))}, \\ \frac{\partial}{\partial \sigma_2} \text{Log}[L(\underline{\Theta})] &= -\frac{2n}{\sigma_2} + \sum_{i=1}^n \frac{y_i^2}{\sigma_2^3} + \sum_{i=1}^n \frac{2e^{-\frac{y_i^2}{2\sigma_2^2}}(\lambda_2 + \lambda_3)y_i^2}{b^3 \left(1 + \left(-1 + 2e^{-\frac{x_i^2}{2\sigma_1^2}} \right) (\lambda_1 + \lambda_3) + \left(-1 + 2e^{-\frac{y_i^2}{2\sigma_2^2}} \right) (\lambda_2 + \lambda_3) \right)}, \\ \frac{\partial}{\partial \lambda_1} \text{Log}[L(\underline{\Theta})] &= \frac{-1 + 2e^{-\frac{x_i^2}{2\sigma_1^2}}}{1 + (-1 + 2e^{-\frac{x_i^2}{2\sigma_1^2}})(\lambda_1 + \lambda_3) + (-1 + 2e^{-\frac{y_i^2}{2\sigma_2^2}})(\lambda_2 + \lambda_3)}, \\ \frac{\partial}{\partial \lambda_2} \text{Log}[L(\underline{\Theta})] &= \frac{-1 + 2e^{-\frac{y_i^2}{2\sigma_2^2}}}{1 + (-1 + 2e^{-\frac{x_i^2}{2\sigma_1^2}})(\lambda_1 + \lambda_3) + (-1 + 2e^{-\frac{y_i^2}{2\sigma_2^2}})(\lambda_2 + \lambda_3)}, \\ \frac{\partial}{\partial \lambda_3} \text{Log}[L(\underline{\Theta})] &= \frac{-2 + 2e^{-\frac{x_i^2}{2\sigma_1^2}} + 2e^{-\frac{y_i^2}{2\sigma_2^2}}}{1 + (-1 + 2e^{-\frac{x_i^2}{2\sigma_1^2}})(\lambda_1 + \lambda_3) + (-1 + 2e^{-\frac{y_i^2}{2\sigma_2^2}})(\lambda_2 + \lambda_3)}. \end{aligned}$$

Since the likelihood equations cannot be solved analytically, a numerical optimization method such as Newton–Raphson, BFGS, or another suitable nonlinear optimization algorithm can be employed. The observed information matrix is defined by the negative Hessian matrix of the log-likelihood: For the five parameters, it has the form

Evaluated at the MLE, the estimated covariance matrix of the parameter estimators is approximated by

$$J(\underline{\theta}) = -\frac{\partial^2 l(\underline{\theta})}{\partial \underline{\theta} \partial \underline{\theta}^T} = - \begin{pmatrix} \frac{\partial^2 \ell}{\partial \sigma_1^2} & \frac{\partial^2 \ell}{\partial \sigma_1 \partial \sigma_2} & \frac{\partial^2 \ell}{\partial \sigma_1 \partial \lambda_1} & \frac{\partial^2 \ell}{\partial \sigma_1 \partial \lambda_2} & \frac{\partial^2 \ell}{\partial \sigma_1 \partial \lambda_3} \\ \frac{\partial^2 \ell}{\partial \sigma_2 \partial \sigma_1} & \frac{\partial^2 \ell}{\partial \sigma_2^2} & \frac{\partial^2 \ell}{\partial \sigma_2 \partial \lambda_1} & \frac{\partial^2 \ell}{\partial \sigma_2 \partial \lambda_2} & \frac{\partial^2 \ell}{\partial \sigma_2 \partial \lambda_3} \\ \frac{\partial^2 \ell}{\partial \lambda_1 \partial \sigma_1} & \frac{\partial^2 \ell}{\partial \lambda_1 \partial \sigma_2} & \frac{\partial^2 \ell}{\partial \lambda_1^2} & \frac{\partial^2 \ell}{\partial \lambda_1 \partial \lambda_2} & \frac{\partial^2 \ell}{\partial \lambda_1 \partial \lambda_3} \\ \frac{\partial^2 \ell}{\partial \lambda_2 \partial \sigma_1} & \frac{\partial^2 \ell}{\partial \lambda_2 \partial \sigma_2} & \frac{\partial^2 \ell}{\partial \lambda_2 \partial \lambda_1} & \frac{\partial^2 \ell}{\partial \lambda_2^2} & \frac{\partial^2 \ell}{\partial \lambda_2 \partial \lambda_3} \\ \frac{\partial^2 \ell}{\partial \lambda_3 \partial \sigma_1} & \frac{\partial^2 \ell}{\partial \lambda_3 \partial \sigma_2} & \frac{\partial^2 \ell}{\partial \lambda_3 \partial \lambda_1} & \frac{\partial^2 \ell}{\partial \lambda_3 \partial \lambda_2} & \frac{\partial^2 \ell}{\partial \lambda_3^2} \end{pmatrix}$$

Under standard regularity conditions, the maximum likelihood estimator $\hat{\underline{\theta}}$ is asymptotically normally distributed:

$$\sqrt{n}(\hat{\underline{\theta}} - \underline{\theta}) \xrightarrow{d} \mathcal{N}_5(\mathbf{0}, J^{-1}(\underline{\theta}))$$

In practice, the inverse of the observed information matrix evaluated at the numerical estimates $\hat{\underline{\theta}}$, denoted as $\widehat{\text{Var}}(\hat{\underline{\theta}}) = J^{-1}(\hat{\underline{\theta}})$, the approximate $(1 - \alpha)100\%$ asymptotic confidence interval for each parameter θ_j is given by:

$$\hat{\underline{\theta}} \pm z_{\alpha/2} \sqrt{\widehat{\text{Var}}(\hat{\underline{\theta}})}$$

where $z_{\alpha/2}$ is the upper $(\alpha/2)$ -th quantile of the standard normal distribution. For instance, the approximate 95% confidence interval is computed as:

$$\hat{\underline{\theta}} \pm 1.96 \sqrt{\widehat{\text{Var}}(\hat{\underline{\theta}})}.$$

5. Simulation Study



To study the performance of the maximum likelihood estimators (MLEs) for the parameters of the Bivariate Transmuted Rayleigh Distribution (BTRD), a comprehensive Monte Carlo simulation study is constructed as follows: The simulation procedure generates $N = 10,000$ independent pseudo-random samples from the BTRD for various sample sizes $n \in \{30, 50, 100, 200\}$. The efficiency and accuracy of the estimators are evaluated across N replicates using two standard statistical criteria: Bias which defined as,

$$\text{Bias}(\hat{\underline{\theta}}) = \frac{1}{N} \sum_{k=1}^N (\hat{\underline{\theta}}^k - \underline{\theta})$$

and mean squared error (MSE):

$$\text{MSE}(\hat{\underline{\theta}}) = \frac{1}{N} \sum_{k=1}^N (\hat{\underline{\theta}}^k - \underline{\theta})^2$$

Also, average confidence interval length (ACL) and coverage probability (CP): The 95% asymptotic confidence intervals are constructed using the observed information matrix, reporting the proportion of iterations where the true parameter falls within the interval ($\text{CP} \approx 0.95$). Results of simulation study are shown in Table 1 as follows:

Table 1: Results of Simulation Study for True value $\underline{\theta} = (\sigma_1, \sigma_2, \lambda_1, \lambda_2, \lambda_3) = (1.2, 0.8, 0.5, -0.4, 0.3)$

Parameter	n	MLE	Bias	MSE	ACL	95% CP
σ_1 σ_2 λ_1 λ_2 λ_3	30	1.2581	0.0581	0.0412	0.7681	0.932
		0.8342	0.0342	0.0189	0.5183	0.938
		0.5612	0.0612	0.0891	1.1245	0.921
		-0.4521	-0.0521	0.0762	1.0348	0.925
		0.3418	0.0418	0.0654	0.9602	0.919
	50	1.2315	0.0315	0.0221	0.5753	0.941
		0.8194	0.0194	0.0098	0.3804	0.945
		0.5341	0.0341	0.0482	0.8415	0.936
		-0.4289	-0.0289	0.0415	0.7761	0.939
		0.3224	0.0224	0.0351	0.7183	0.931
	100	1.1242	0.0142	0.0104	0.3951	0.948
		0.8085	0.0085	0.0045	0.2601	0.951
		0.5162	0.0162	0.0215	0.5682	0.944
		-0.4135	-0.0135	0.0189	0.5310	0.947
		0.3098	0.0098	0.0162	0.4912	0.943
	200	1.2051	0.0051	0.0048	0.2704	0.950
		0.8031	0.0031	0.0021	0.1782	0.952
		0.5068	0.0068	0.0098	0.3841	0.949
		-0.4052	-0.0052	0.0086	0.3592	0.951
		0.3039	0.0039	0.0074	0.3325	0.948

The simulation results show the finite-sample accuracy and asymptotic efficiency of the maximum likelihood estimators for the BTRD parameters $(\sigma_1, \sigma_2, \lambda_1, \lambda_2, \lambda_3)$. As the sample size increases from $n = 30$ to $n = 200$, both the absolute bias and mean squared error (MSE) for all parameters decay monotonically toward zero, confirming estimator consistency. For coverage probabilities (CP) converge rapidly to the nominal 95% level as $n \geq 50$. Overall, the results validate that the proposed numerical MLE framework and asymptotic normal approximation yield precise inferences in moderate to large samples.

6. Real Applications and Comparative Study with Other Distributions

In this section, we present two real applications of BTRD to demonstrate its effectiveness in practical scenarios. We compare it with other distributions to illustrate that the performance of BTRD surpasses that of the alternatives, utilizing the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) for evaluation. According to these criteria, a distribution is considered superior if it has the lowest value of AIC or BIC.

6.1 Economic Application

[20] introduced economic data, emphasizing its significance across all countries. The data encompasses exports of goods and services (X) and GDP growth (Y). It consists of 31 bivariate observations from 1980 to 2010. Table 2 presents the data, Table 3 provides the descriptive statistics, and Table 4 compares BTRD with other distributions.

Table 2: Economic Data

Years	X	Y	Years	X	Y
1980	30.51	10.01	1996	20.75	4.99
1981	33.37	3.76	1997	18.84	5.49
1982	27.03	9.91	1998	16.21	4.04



1983	25.48	7.4	1999	15.05	6.11
1984	22.35	6.09	2000	16.2	5.37
1985	19.91	6.6	2001	17.48	3.54
1986	15.73	2.65	2002	18.32	2.37
1987	12.56	2.52	2003	21.8	3.19
1988	17.32	7.93	2004	28.23	4.09
1989	17.89	4.97	2005	30.34	4.48
1990	20.05	5.7	2006	29.95	6.85
1991	27.82	1.08	2007	30.25	7.09
1992	28.4	4.43	2008	33.04	7.16
1993	25.84	2.9	2009	24.96	4.67
1994	22.57	3.97	2010	21.35	5.15
1995	22.55	4.64			

Table 3: Descriptive Statistics

Statistics	X	Y
min	12.56	1.08
max	33.37	10.01
mean	22.95	5.13
median	22.35	4.97
Standard deviation	1.037	2.08
Pearson's correlation	0.2657	
Kendall's tau correlation	0.2008	
Spearman's rho correlation	0.1311	

Table 4: Comparative Study between BTRD and Other Distributions

Distribution	BTRD	FGMBF	AMHBF
Maximum likelihood estimators	$\hat{\sigma}_1 = 20.7409$ $\hat{\sigma}_2 = 4.5400$ $\hat{\lambda}_1 = -0.2347$ $\hat{\lambda}_2 = -0.1573$ $\hat{\lambda}_3 = 0.2201$	$\hat{\alpha}_1 = 19.4831$ $\hat{\lambda}_1 = 3.9925$ $\hat{\alpha}_2 = 3.6548$ $\hat{\lambda}_2 = 1.7968$ $\hat{\theta} = 0.7862$	$\hat{\alpha}_1 = 19.3297$ $\hat{\lambda}_1 = 3.9772$ $\hat{\alpha}_2 = 3.6783$ $\hat{\lambda}_2 = 1.7986$ $\hat{\theta} = 0.6615$
AIC	245.1249	358.2544	357.9778
BIC	252.2949	360.5916	360.3150

*Farlie-Gumbel- Morgenstern bivariate Fréchet (FGMBF)

**Ali-Mikhail-Haq Bivariate Fréchet (AMHBF)

First, the goodness-of-fit of the proposed BTRD model is assessed using the bivariate Kolmogorov–Smirnov (K–S) test and the Cramér–von Mises (C–vM) distance. The bivariate K–S test yields a statistic of 0.1147 with a p-value of 0.428, while the C–vM test gives a statistic of 0.0618 with a p-value of 0.512. Since both p-values are greater than the conventional significance level of, there is insufficient evidence to reject the null hypothesis that the proposed BTRD model provides an adequate fit to the data. Therefore, the results support the suitability of the BTRD model for this data. Furthermore, as shown in Table 4, the proposed BTRD model has the smallest values of both AIC and BIC compared with the two competing distributions. Since lower AIC and BIC values indicate a better balance between goodness of fit and model complexity, these results provide additional evidence that BTRD provides the best overall fit among the three considered models.

6.2 The UEFA Champion's League Data

[21] introduced the UEFA Champions League data set. This data is bivariate, where X represents the time (in minutes) of the first goal scored by any team, and Y represents the time of the first goal of any type scored by the home team. Table 5 presents the data, Table 6 provides the descriptive statistics, and Table 7 presents a comparative analysis of the proposed BTRD model and the competing distributions, where the observed failure times were standardized by applying the transformations $x^* = \frac{x}{100}$ and $y^* = \frac{y}{100}$ prior to parameter estimation. This scaling facilitates numerical optimization and ensures computational stability.

Table 5: UEFA Champion's League Data for 2004–2006

2005-2006		2004-2005	
X	Y	X	Y
26	20	34	34
63	18	53	39



19	19	54	7
66	85	51	28
40	40	76	64
49	49	64	15
8	8	26	48
69	71	16	16
39	39	44	13
82	48	25	14
72	72	55	11
66	62	49	49
25	9	24	24
41	3	44	30
16	75	42	3
18	18	27	47
22	14	28	28
42	42	2	2
36	52		

Table 6: Descriptive Statistics

Statistics	X	Y
min	2	2
max	82	14
mean	40.89	32.86
median	41	22.5222
Standard deviation	19.8641	
Pearson's correlation	0.4698	
Kendall's tau correlation	0.4074	
Spearman's rho correlation	0.3110	

Table 7: Comparative Study between BTRD and Other Distributions

Distribution	BTRD	BGLFR	BGLE
Maximum likelihood estimators	$\hat{\sigma}_1 = 0.3277$ $\hat{\sigma}_2 = 0.3028$ $\hat{\lambda}_1 = 0.2093$ $\hat{\lambda}_2 = 0.3411$ $\hat{\lambda}_3 = 0.1731$	$\hat{\alpha}_1 = 0.4920$ $\hat{\alpha}_2 = 0.1660$ $\hat{\alpha}_3 = 0.4110$ $\hat{\beta} = 0.0002$ $\hat{\gamma} = 0.0008$	$\hat{a}_1 = 0.00001$ $\hat{b}_1 = 0.0007$ $\hat{a}_2 = 0.0031$ $\hat{b}_2 = 0.0009$ $\hat{\theta} = 0.7590$
AIC	399.69	596.75	657.40
BIC	407.75	604.81	665.45

* Bivariate Generalized Linear Exponential (BGLE)

** Bivariate Generalized Linear Failure Rate (BGLFR)

First, the goodness-of-fit of the proposed BTRD model is assessed using the bivariate Kolmogorov–Smirnov (K–S) test and the Cramér–von Mises (C–vM) distance. The bivariate K–S test yields a statistic of 0.1381 with a p-value of 0.492, while the C–vM test gives a statistic of 0.0762 with a p-value of 0.546. Since both p-values are greater than the conventional significance level of, there is insufficient evidence to reject the null hypothesis that the proposed BTRD model provides an adequate fit to the data. Therefore, the results support the suitability of the BTRD model for this data.

Furthermore, as shown in Table 7, the proposed BTRD model has the smallest values of both AIC and BIC compared with the two competing distributions. Since lower AIC and BIC values indicate a better balance between goodness of fit and model complexity, these results provide additional evidence that BTRD provides the best overall fit among the three considered models, see [22].

7. Conclusions

In this paper, we investigated a new bivariate Transmuted Rayleigh distribution (BTRD) for modeling dependent bivariate data. The proposed model provides a flexible framework for describing dependence between two random variables while retaining the desirable characteristics of the Transmuted Rayleigh distribution in the marginal distributions. Some important statistical properties of the proposed distribution were derived. The model parameters were estimated using the maximum likelihood method. Since the resulting likelihood equations do not generally admit closed-form solutions, numerical optimization techniques were employed to obtain the maximum likelihood estimates. In addition, the asymptotic properties of the estimators were investigated through the observed information matrix, which provides the basis for estimating standard errors and constructing asymptotic confidence intervals. A comprehensive simulation study



was conducted to examine the performance of the maximum likelihood estimators under different sample sizes. The simulation results indicate that the proposed estimation procedure performs satisfactorily, with the estimates. The real applications of the proposed model were demonstrated through two real-data applications involving dependent bivariate observations. The BTRD was compared with several established competing bivariate models, including FGMBF, AMHBF, BGLE, and BGLFR, using standard model-selection criteria such as the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). The empirical results demonstrate that the proposed BTRD provides a competitive. Future research could extend the proposed BTRD to higher dimensions and investigate alternative dependence structures and estimation methods, including Bayesian. Further work could also consider censored and truncated observation.

References

- [1] M. Bourguignon, I. Ghosh, and G. M. Cordeiro, "General results for the transmuted family of distributions and new models," *Journal of Probability and Statistics*, 2016.
- [2] F. Merovci, M. Alizadeh and G. G. Hamedani, "Another generalized transmuted family of distributions: properties and applications," *Austrian Journal of Statistics*, vol. 45, no. 3, pp. 71-93, 2016.
- [3] M. Alizadeh, F. Merovci and G. G. Hamedani, "Generalized transmuted family of distributions: properties and applications," *Hacettepe Journal of Mathematics and Statistics*, vol. 46, pp. 645-667, 2017.
- [4] F. Merovci, M. Alizadeh, H. M. Yousof and G. G. Hamedani, "The exponentiated transmuted-g family of distributions: theory and applications," *Communications in Statistics - Theory and Methods*, vol. 46, no. 21, pp. 10800-10822, 2017.
- [5] S. Rezaei, A. K. Marvasty, S. Nadarajah and M. A. Alizadeh, "A new exponentiated class of distributions: Properties and applications," *Communications in Statistics - Theory and Methods*, vol. 46, no. 12, pp. 6054-6073, 2017.
- [6] H. S. Bakouch, F. Jamal, C. Chesneau and A. Nasir, "A new transmuted family of distributions: Properties and estimation with applications," *Journal of Statistics and Management Systems*, vol. 26, no. 2, pp. 261-289, 2023.
- [7] M. Yilmaz and H. Ünözkan, "On bivariate extension of the univariate transmuted distribution family," *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, vol. 70, no. 2, pp. 1055-1064, 2021.
- [8] J. A. Darwish, L. I. Al turk and M. Q. Shahbaz, "The Bivariate Transmuted Family of Distributions: Theory and Applications," *Computer Systems Science and Engineering*, vol. 36, no. 1, pp. 131-144, 2021.
- [9] F. Merovci, "Transmuted Rayleigh Distribution," *AUSTRIAN JOURNAL OF STATISTICS*, vol. 42, no. 1, pp. 21-31, 2013.
- [10] N. L. Johnson, S. Kotz and N. Balakrishnan, *Continuous Univariate Distributions*, 2 ed., New York: John Wiley & Sons Ltd, 1994.
- [11] D. F. Moore, *Applied Survival Analysis Using R*, Cham: Springer, 2016.
- [12] M. Modarres, M. P. Kaminskiy and V. Krivtsov, *Reliability Engineering and Risk Analysis: A Practical Guide*, third ed., UK: Taylor & Francis Group, 2017.
- [13] M. Hassan and C. Chesneau, "Bivariate Generalized Half-Logistic Distribution: Properties and Its Application in Household Financial Affordability in KSA," *Mathematical and Computational Applications*, vol. 27, no. 4, 2022.
- [14] P. W. Holland and Y. J. Wang, "Dependence function for continuous bivariate densities," *Communications in Statistics - Theory and Methods*, vol. 16, no. 3, pp. 863-876, 1987.
- [15] N. Balakrishnan and C. D. Lai, *Continuous bivariate distributions*, 2 ed., New York: Springer, 2009.
- [16] D. G. Clayton, "A model for association in bivariate life tables and its application in epidemiological studies of familial tendency in chronic disease incidence," *Biometrika*, vol. 65, no. 1, pp. 141-151, 1978.
- [17] D. Oakes, "Bivariate survival models induced by frailties," *Journal of the American Statistical*, vol. 84, pp. 487-493, 1989.
- [18] R. C. Gupta, "On some association measures in bivariate distributions and their relationships," *Journal of Statistical Planning and Inference*, vol. 117, no. 1, pp. 83-93, 2003.
- [19] J. E. Anderson, T. A. Louis, N. V. Holm and B. Harvald, "Time-dependent association measures for bivariate survival distributions," *Journal of the American Statistical Association*, vol. 87, no. 419, pp. 641-650, 1992.
- [20] E. M. Almetwally and H. Z. Muhammed, "On a Bivariate Fréchet Distribution," *Journal of Statistics Applications & Probability*, vol. 9, no. 1, pp. 71-91, 2020.
- [21] S. G. Meintanis, "Test of fit for Marshall-Olkin distributions with applications," *Journal of Statistical Planning and Inference*, vol. 137, no. 12, pp. 3954-3963, 2007.
- [22] A. K. Pathak and P. Vellaisamy, "A bivariate generalized linear exponential distribution: properties and estimation," *Communications in Statistics - Simulation and Computation*, vol. 51, no. 9, pp. 5426-5446, 2022.

Declaration

Conflict of Study: The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Funding Statement: This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Author Contribution Statement: Marwa KH. Hassan conceived the idea and designed the research; Analyzed and interpreted the data and wrote the paper.



Acknowledgment: Use of ChatGPT for English improvements

Availability of Data and Material: Data will be made available by the corresponding author on reasonable request.

Author(s) Bio / Authors' Note

***Marwa KH. Hassan** has been involved in research and teaching for various National and International institutes for more than three years. He is the author of one book and a contributor/associate editor in two others, along with different high-ranking journal articles; he also participated in various reputed international conferences.*

